Competing auctions with informed sellers*

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Abstract

We study competing auctions where each seller has private information about the quality of his object and chooses the reserve price of a second-price auction. Buyers observe the reserve prices and decide which auction to participate in. For a class of primitives, we show that a perfect Bayesian equilibrium exists for any finite market. In any such PBE, higher quality is signaled through higher reserve price at the expense of trade opportunities. But there might be bunching regions causing inefficiencies. In fact, in the large-market limit characterized by a directed search model, the interaction of adverse selection and search frictions entail distortion at the bottom: when either the buyer-seller ratio is sufficiently large or a regularity condition is met, there is no separating PBE in which the lowest-quality seller sets reserve price equal to his opportunity cost. This finding carries over to large finite markets and is consistent with observed behavior in auctions for used cars in UK (Choi, Nesheim and Rasul, 2016).

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1 Introduction

Many important economic activities are operated through competing auctions with privately informed sellers. For instance, some used-car markets operate with competing sellers soliciting bids from buyers, and the sellers are better informed than the buyers about the quality of their cars. Similarly, there are competing auctions for artwork and real estate, where sellers solicit bids from buyers and are better informed about the quality of the objects. Also we see a plethora of online auction sites like eBay, where competing sellers solicit bids for objects about whose qualities the sellers are better informed. In spite of all these important examples, the literature has not yet provided a study of competing auctions with informed sellers who sell differentiated objects. The objective of this paper is to provide a first such study.

We focus on competing second-price auctions with reserve price as follows. Each seller has private information about the quality of his object; and he chooses the reserve price of his auction. Each buyer has private information about her valuation of objects with different qualities. And after observing the posted reserve prices, she decides to participate in at most one auction.[1] Finally, each auction is implemented for its participants.

Observe that there are adverse selection from the seller's private information and search frictions from the buyer's side. Without adverse selection, it is well-known that in a large market the revenue-maximizing reserve price equals the opportunity cost, and hence decentralization is efficient subject to the constraint of search frictions.[2] But, with adverse selection, sellers can use reserve prices to signal their private information, so reserve prices have an additional revenue impact and it is not clear that constrained efficiency will prevail. In fact, one take away of our results is that inefficiencies due to bunching are inevitable.

Moving to the analysis, we first study a model with finitely many sellers and buyers. We assume that (i) a higher-quality seller has a higher opportunity cost and (ii) a buyer's valuation of an object is additively separable in her own type and the quality of the object. Based on Hernando-Veciana (2005), buyers' participation/search behavior is uniquely characterized conditional on the posted reserve prices and the perceived qualities of objects. However, when the space of reserve prices is a continuum, existence of perfect Bayesian equilibrium (PBE) has not being established in the literature. In fact, the

[1] We adopt this assumption to avoid the complications of unit demand buyers winning multiple objects.
[2] See McAfee (1993), Peters and Severinov (1997), Hernando-Veciana (2005) and Virag (2010).

(virtual) valuation of a buyer can be endogenously discontinuous and non-monotonic, so the existing techniques (e.g. the differential approach of Mailath (1987) and Mailath (1988)) are not applicable. Our first main result is to establish existence of a PBE. We do so by employing a finite approximation technique and by establishing that the discontinuity of Bayesian updating can be overcome under our assumptions. Also, we obtain that in any PBE, higher quality is signaled through higher reserve price at the expense of trade opportunities: a seller with higher quality posts a weakly higher reserve price and trades with weakly lower probability.

Due to the complications generated by signaling and endogenous buyer behavior, further analysis of finite markets seems to be difficult. To achieve deeper understanding, we formulate a directed search environment which is the limit of our finite-market model as the number of players on both sides of the market increase at a constant buyer-seller ratio $k > 0$ as in Peters and Severinov (1997). When the buyer-seller ratio is sufficiently large or if a regularity condition is imposed, we find that there is no separating limit equilibrium in which the lowest-quality seller sets reserve price equal to his opportunity cost.[3] In other words, any limit equilibrium has a distortion at the lower end of the quality distribution: either (1) the lowest-quality seller sets a reserve price strictly higher than his opportunity cost, which implies that there are mutually beneficial trade opportunities missing; or (2) a positive measure of seller types at the lower end post the same reserve price, which impairs informativeness and direction of buyers' search behavior. Intuitively, separation of higher types from lower types requires sufficient variations in trade opportunities, but the demand side at the lower end of the market could be extremely thick under quality separation, which locally goes against separation. These distortions found in the large markets carry over to large finite markets for the class of primitives under consideration, and vanishing distortions in large finite markets are impossible.

We also show that the first type of distortion described above fails a naive robustness check, and the second type of distortion may be a better prediction of equilibrium behavior. The second type of distortion has a testable implication: at the lower end of the market, reserve prices do not respond to quality variations. This seems to be consistent with a study of the used car auction market in the UK conducted by Choi, Nesheim and Rasul (2016). They document that for some categories of used cars, the reserve price is

[3]When each seller's type and each buyer's type follow the uniform distribution on the unit interval, the corresponding cutoff buyer-seller ratio is $\frac{1}{2}$: as long as the number of buyers is more than half of the number of sellers, there is no separating limit equilibrium.

constant in privately assessed quality when the quality is sufficiently low.[4]

The rest of the paper is organized as follows. Section 2 discusses related literature. Section 3 introduces the model for finite markets, characterizes buyers' behavior, and establishes monotonicity and existence of a symmetric PBE. Section 4 considers large markets, and studies the impossibility of fully separating seller behavior. Section 5 concludes.

2 Literature Review

Competing second-price auctions in which sellers have homogeneous objects to trade is well-studied. Peters and Severinov (1997) set up a large-market model which has a finite-market foundation with sellers playing pure strategies, formulate the concept of competitive matching equilibrium, and show that in a competitive matching equilibrium, each seller posts a reserve price equal to his opportunity cost. Our large-market analysis is inspired by their approach. For finite markets, Burguet and Sakovics (1999) considers the two-seller case. They fully characterize buyers' behavior and establish non-existence of pure strategy equilibrium. They also show that in equilibrium, each seller posts a reserve price strictly higher than his opportunity cost. Bridging the gap between Peters and Severinov (1997) and Burguet and Sakovics (1999), Hernando-Veciana (2005) shows that when the space of reserve price is finite, equilibrium reserve prices are driven by competition to opportunity costs in a sufficiently large finite market; Virag (2010) later shows that when the space of reserve price is a continuum, equilibrium reserve prices converge to opportunity costs when market size grows. We stress that the elegant characterization of buyers' behavior in Hernando-Veciana (2005) is helpful for our monotonicity and existence results in finite markets. Allowing for general direct mechanisms, McAfee (1993) considers competitive equilibria in finite markets, and shows that there is an equilibrium in which each seller offers a second-price auction with reserve price equal to opportunity cost; also Eeckhout and Kircher (2010) show in a directed search environment that when the meeting technology is non-rival, only auction-like mechanisms can arise in equilibrium. These may stand as support for our focus on second-price auctions with reserve price.

[4] More details about the reserve pricing rule of an auctioneer are documented by these authors: as a function of the privately assessed quality, the reserve price is constant up to a cutoff, jumps (discontinuously) upward at the cutoff, and is increasing thereafter. This rule is consistent with a semi-pooling equilibrium of our model. However, we don't have a full characterization of such equilibria. See also our Remark 4.3.

Cai, Riley and Ye (2007) study a model with a single informed seller signaling quality by reserve price. They characterize a separating equilibrium as the solution of an ordinary differential equation, and establish existence and uniqueness under a single crossing assumption on virtual valuation[5]. Also in the separating equilibrium, the lowest seller type posts his full-information optimal reserve price (opportunity cost), and in this sense there is no distortion at the lower end of the market. Mailath (1987) and Mailath (1988) study separating equilibrium in general signaling games also with a differential approach and single crossing assumptions. As discussed in the Introduction, the strategic interaction in our model generates difficulty for ensuring single crossing properties by assumptions on primitives. Therefore, we adopt a finite-approximation approach to establish existence of equilibrium. Our finite-approximation step is similar in spirit to the discussion in the online appendix of Mensch (2017), but it is not clear whether his assumptions apply to our model. Kartik (2009) studies a model of communication with lying costs where one principal has a bounded type space and shows non-existence of separating equilibrium. Intuitively, separation requires sufficient variation in possible lies, which is impossible due to boundedness of the type space. In some sense, our non-existence result is also affected by the "boundedness" of the probability of trade as quality separation requires sufficient variation of trade opportunities. However, in our context, the probability of trade is an endogenous object that depends on the competition between sellers and the participation of buyers.

Markets with multiple informed sellers has received growing attention in the literature of directed search. Guerrieri, Shimer and Wright (2010) build a general and useful framework to analyze adverse selection and search friction when one side of the market is uninformed and meeting is bilateral. They establish existence and payoff equivalence of separating equilibria. Kim and Kircher (2015) study a model in which informed sellers with private values offer first-price auctions with zero reserve price and communicate information to buyers by cheap talk. They show that a fully informative equilibrium exists and constrained efficiency is achieved in this equilibrium. They also point out the importance of auction format and show that informative cheap talk is impossible with second-price auctions. These two insightful papers leave open questions about environments in which informed sellers not only possess information that is payoff-relevant for buyers but can also choose and commit to different mechanisms that allow meeting with multiple buyers. Our paper contributes to the understanding of these environ-

[5] Jullien and Mariotti (2006) obtain similar results.

ments with a focus on second-price auctions with reserve price, and we show that full informativeness may be impossible. Delacroix and Shi (2013) study a model in which sellers can signal quality by posted price. In spite of important difference in many aspects compared to our model, they also highlight the tension between directing search and signaling.

3 Finite market

3.1 Finite-market model

We first set up a game of competing second-price auctions with informed sellers in finite markets. There is a finite set of sellers J with $|J| \geq 2$. Each seller $j \in J$ has an object with quality $s_j \in S = [\underline{s}, \bar{s}]$ to sell in a second-price auction, and he[6] chooses the reserve price $r_i \in R = [\underline{r}, \bar{r}]$ of his auction. We assume that $\bar{r}$ is high enough to preclude trade and $\underline{r}$ is low enough to guarantee trade. We also refer to J as the set of objects. There are $k|J|$ buyers. Each buyer i has a type $\theta_i \in \Theta = [0, 1]$,[7] and she can bid in at most one auction.

At the beginning of the game, the quality of each object is drawn i.i.d from the cumulative distribution $G : S \to [0, 1]$, and the type of each buyer is drawn i.i.d from the cumulative distribution $F : \Theta \to [0, 1]$. Then sellers choose simultaneously reserve prices. After observing the posted reserve prices, buyers choose simultaneously at most one auction to participate in. We assume that qualities (s_j) and types (θ_i) are private information, while G and F are common knowledge. Finally, each auction is implemented for its participants. Note that with second-price auctions, each buyer bids her true valuation in the undominated equilibrium regardless of whether she observes the participation decision of other buyers.[8]

If object j is sold to buyer i at price t, the payoff to seller j is t, and the payoff to buyer i is $\alpha(\theta_i) + \beta(s_j) - t$. If seller j fails to trade, he obtains his opportunity cost $c(s_j)$; if buyer i fails to trade, she obtains 0. Note that the value of an object for a buyer is additively separable in buyer type and object quality. This assumption is crucial to the participation/search behavior of buyers, and consequently it is also crucial to the main results we will present. In Appendix D, we characterize buyers' behavior for

[6]We use the pronouns he for a seller and she for a buyer.

[7]Buyer's type can be interpreted as her taste for objects and is unrelated to object qualities.

[8]As a consequence of revenue equivalence theorem, our results still hold for some other auction formats
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a larger class of buyer's valuations in a market with two sellers, and we discuss the corresponding implications.

We also impose the following assumptions on the primitives described above:

1. G has a positive density g over S, and F has a postive density f over Θ.

2. g, f, α, β and c are $\mathcal{C}^1$.

3. The first derivatives α', β' and c' are positive.

Note that as both β and c are strictly increasing, a seller who has a higher quality object also has a higher opportunity cost, and thus he suffers less from failure of trade. This creates the possibility of signaling quality through sacrificing trade opportunities. And to understand how signaling takes place in equilibrium, it is important to understand how the choice of reserve price of a seller affects his probability of trade.

The solution concept adopted for the finite-market model is symmetric perfect Bayesian equilibrium (PBE). In particular, each seller adopts the same strategy $\rho : S \to \Delta(R)$, and belief updating scheme is represented by $\Lambda : R \to \Delta(S)$. Assuming that each buyer always bids her true valuation, we focus on their participation/search strategy which is represented by $p : \Theta \times R^J \times \Delta(S)^J \to \Delta(J)$. For each $j \in J$ and each $(\boldsymbol{r}, \boldsymbol{\lambda}) \in R^J \times \Delta(S)^J$, let $V_\theta(j; p, \boldsymbol{r}, \boldsymbol{\lambda})$ be the interim payoff of a type θ buyer participating in auction j, when all other buyers play p, and the profile of reserve prices and perceived qualities is $(\boldsymbol{r}, \boldsymbol{\lambda})$. For each $s \in S$, let $u_s(r; \rho, \Lambda, p)$ be the interim payoff of a type s seller who posts r as reserve price, when all other sellers play ρ, all buyers update their beliefs over qualities according to Λ and play p. Then (ρ, Λ, p) is a symmetric PBE if for each $s \in S$, each $\theta \in \Theta$ and each $(\boldsymbol{r}, \boldsymbol{\lambda}) \in R^J \times \Delta(S)^J$:

$$\rho(s)(\arg\max_{r \in R} u_s(r; \rho, \Lambda, p)) = 1, \qquad \text{(Seller optimality)}$$

$$p(\theta, \boldsymbol{r}, \boldsymbol{\lambda})(\arg\max_{j \in J} V_\theta(j; p, \boldsymbol{r}, \boldsymbol{\lambda})) = 1, \qquad \text{(Buyer optimality)}$$

and Λ is consistent with Bayesian updating with respect to ρ.[9]

[9]Note that due to discontinuities generated by belief updating or response of buyers, the maximization problem of a seller is not well defined in general. However, our analysis below shows that such discontinuities can be overcome in equilibrium, and we simply present seller optimality with "arg max".

3.2 Buyers' behavior

Suppose after observing a profile of reserve prices $(r_j)_{j\in J}$, buyers form beliefs about object quality (perceived quality) as $(\lambda_j)_{j\in J}$. Given a pair (r_j, λ_j), let m_j be the unique solution to

$$\alpha(m_j) + \int \beta(s)\lambda_j(ds) = r_j,$$

i.e., m_j is the minimum type of a buyer willing to win object j at its reserve price r_j. Following directly from Lemma 1 of Hernando-Veciana (2005), the following proposition establishes that the profile of minimum types $\boldsymbol{m} = (m_j)_{j\in J}$ completely determines the participation/search behavior of buyers.

Proposition 3.1. Fix $m_1 \leq m_2 \leq ... \leq m_{|J|}$. There is a unique profile of cutoff participation types $\boldsymbol{t} \in \Theta^J$ with $t_1 \leq t_2 \leq ... \leq t_{|J|} \leq t_{|J|+1} = 1$ such that in the unique symmetric equilibrium of the participation game parametrized by $\boldsymbol{m}$, a buyer with type $\theta \in [t_j, t_{j+1}]$ participates in the j lowest-minimum-type auctions each with probability $1/j$.

Observe that when a buyer of type θ wins object j at its reserve price, her payoff is $\alpha(\theta) - \alpha(m_j)$; if she wins at the second highest bid, additive separability of buyer type and object quality again cancels β, the direct utility effect of quality. Then in terms of buyers' participation/search behavior, our model with heterogeneous objects is equivalent to the well-studied model with homogeneous objects in which the "reserve price" of auction j is $\alpha(m_j)$, and we omit the proof of Proposition 3.1. Although quality affects buyers' participation/search decision only through induced minimum type, it should be noticed that when (r, λ) and (r', λ') induce the same minimum type, the bid submitted by a participant varies with perceived quality.

Given the characterization in Proposition 3.1, let

$$q_j = 1 - \sum_{j' \geq j} \frac{F(t_{j'+1}) - F(t_{j'})}{j'}$$

be the probability that a buyer does *not* participate in auction j. Then the probability that seller j fails to trade is simply $(q_j)^{k|J|}$. As we mentioned above, it is important to understand how choice of reserve price affects probability of no trade. And given Proposition 3.1, it suffices to analyze the effect of induced minimum types. Intuitively, if seller j increases his induced minimum type, he should face a higher probability of no trade. Setting $m_0 = 0$ and $m_{|J|+1} = 1$, we confirm this intuition in the next proposition.

Proposition 3.2. Fix $j \in J$. Suppose $m_j \in (m_{j-1}, m_{j+1})$, then $\frac{\partial q_j}{\partial m_j} > 0$.

The proof is in Appendix A. The main idea of the proof is to exploit the monotonicity of the system defining the mapping from $\boldsymbol{m}$ to $\boldsymbol{t}$ in Hernando-Veciana (2005) and to show that it suffices to study the property of two (instead of $|J|$) equations.

3.3 Existence of symmetric PBE

Given the characterization of buyers' behavior, we proceed to analyze the strategic interaction between sellers. First we show that in any symmetric PBE, higher on-path reserve price signals higher quality and induces higher probability of no trade; and a seller with higher quality chooses a weakly higher reserve price. Then we show that a symmetric PBE indeed exists by a finite-approximation approach.

Let p^* represent the buyers' behavior described in Proposition 3.1. We will say that (ρ, Λ) constitutes an equilibrium when (ρ, Λ, p^*) is a symmetric PBE. As we need to vary seller's action (reserve price) space and type (quality) space in the finite-approximation argument, we denote the game with reserve price space X and quality space Y as $\Gamma(X, Y)$.

For each $(\boldsymbol{r}, \boldsymbol{\lambda})$, let $u_{j,s}(\boldsymbol{r}, \boldsymbol{\lambda})$ be the payoff to seller j with quality s, taking into account of the buyers' equilibrium response p^*. It is easy to see that $u_{j,s}$ is continuous. Fix (ρ, Λ), we denote payoff to (j, s) from playing r as $u_s(r; \rho, \Lambda)$ (j is omitted from notations by symmetry).[10] Note that $u_s(r; \rho, \Lambda)$ can be written as

$$u_s(r; \rho, \Lambda) = c(s)Q(r; \rho, \Lambda) + \Pi(r; \rho, \Lambda)$$

in which $Q(r; \rho, \Lambda)$ is the probability of no trade and $\Pi(r; \rho, \Lambda)$ is the expected revenue from trade. The fact that quality affects buyers' participation/search behavior only through induced minimum types implies a strong form of monotonicity of seller's strategy.

Proposition 3.3. Suppose (ρ, Λ) constitutes an equilibrium, $r > r'$ are both on-path, and *min* $\{Q(r; \rho, \Lambda), Q(r'; \rho, \Lambda)\} < 1$. Then

(1) $\inf \text{supp}(\Lambda(r)) \geq \sup \text{supp}(\Lambda(r'))$;

(2) $Q(r; \rho, \Lambda) > Q(r'; \rho, \Lambda)$.

[10] Each (ρ, Λ) induces naturally a probability measure ξ_{-j} over $R^{J \setminus \{j\}} \times \Delta(S)^{J \setminus \{j\}}$. And $u_s(r; \rho, \Lambda)$ is defined as the expectation of $u_{j,s}(r, \Lambda(r); \boldsymbol{r}_{-j}, \boldsymbol{\lambda}_{-j})$ with respect to ξ_{-j}.

Proof Since (ρ, Λ) is fixed throughout the proof, we omit it from notations. First we show that (2) implies (1). Suppose towards a contradiction, there are $s' > s$ such that $s \in \text{supp}(\Lambda(r))$ and $s' \in \text{supp}(\Lambda(r'))$. By incentive compatibility,

$$c(s)Q(r) + \Pi(r) \geq c(s)Q(r') + \Pi(r')$$
$$c(s')Q(r) + \Pi(r) \leq c(s')Q(r') + \Pi(r'),$$

which implies

$$c(s) \geq -\frac{\Pi(r) - \Pi(r')}{Q(r) - Q(r')} \geq c(s')$$

and contradicts $c(s') > c(s)$.

Second we show that $r > r'$ implies (2). Suppose towards a contradiction, $Q(r) \leq Q(r')$. If $Q(r) < Q(r')$, then $\inf \text{supp}(\Lambda(r')) \geq \sup \text{supp}(\Lambda(r))$. Let m be the minimum type induced by $(r, \Lambda(r))$, and let m' be the minimum type induced by $(r', \Lambda(r'))$. Obviously, $m > m'$. but then by Proposition 3.2, $Q(r) > Q(r')$, which is a contradiction. If $Q(r) = Q(r')$, then $m = m'$. By Proposition 3.1, r and r' induce identical buyers' participation/search behavior, and revenue increases in perceived quality in a simple way: for each $s \in S$,

$$u_s(r) - u_s(r') = (\beta(\Lambda(r)) - \beta(\Lambda(r')))(1 - Q(r)) > 0,$$

i.e., r' is dominated by r for all seller types, which is a contradiction.

Given sellers' strategy ρ, let $\text{supp}(\rho)$ be the corresponding collection of on-path reserve prices under the prior G. We say that (ρ, Λ) satisfies incentive compatibility on $\text{supp}(\rho)$ if for each $s \in S$, a type s seller has no profitable deviations to reserve prices in $\text{supp}(\rho)$. Note that Proposition 3.3 (1) implies that if (ρ, Λ) satisfies incentive compatibility on $\text{supp}(\rho)$, it is essentially a pure strategy and can be represented by a non-decreasing function from S to R. Then ρ is in a simpler space to work with. Moreover, it also implies a property that is important for the validity of a finite-approximation approach to existence, which we record in the next lemma.

Lemma 3.4. Suppose (ρ, Λ) satisfies incentive compatibility on supp (ρ), $r_1 > r_2$ are both on-path, $s_1 \in \text{supp}(\Lambda(r_1))$ and $s_2 \in \text{supp}(\Lambda(r_2))$. Then there is $D(s_1, s_2) > 0$ such that $r_1 - r_2 > D(s_1, s_2)$. Moreover, $D(s_1, s_2)$ is continuous, decreasing in s_1 and increasing in s_2.

The proof is in the appendix. This lemma says that if two seller-types do not pool, then they separate at reserve prices with a positive distance $D(s_1, s_2)$ independent of at which reserve prices they separate. This property then helps to overcome the general discontinuity of Bayesian updating with respect to weak convergence of strategies. For example, suppose for each n, ρ_n satisfies incentive compatibility on $\text{supp}(\rho_n)$, and $s_1 > s_2$ separate at r_1 and $r_{2,n}$. If ρ_n converges to ρ, it is possible that $\lim_{n\to\infty} r_{2,n} = r_1$, which implies that s_1 and s_2 pool in the limit and entails a discontinuity of Bayesian updating at r_1. However, in our model, Lemma 3.4 implies that $r_1 - \lim r_{2,n} > D(s_1, s_2)$, and thus s_1 and s_2 also separate in the limit.

Given Proposition 3.3 and Lemma 3.4, we obtain our first main result establishing existence of a symmetric PBE in any finite market.

Theorem 1. $\Gamma(R, S)$ has a symmetric PBE.

The idea of the proof is relatively straightforward. Here we provide a sketch and details are in Appendix B. We first show that if the action space of each seller is restricted to a finite subset R_n of R, then $\Gamma(R_n, S)$ has a symmetric PBE (ρ_n, Λ_n). In this step, a key argument is that when the action space is finite, Bayesian updating (viewed as a correspondence from R_n to $\Delta(S)$) is u.h.c with respect to weak convergence of strategies.[11] Second, we consider a sequence of (R_n) approximating R in Hausdorff topology on closed sets, and select an appropriate accumulation point of $((\rho_n, \Lambda_n))$. Each (ρ_n, Λ_n) constitutes an equilibrium in $\Gamma(R_n, S)$, and each ρ_n is regarded as a non-decreasing function from S to R by Proposition 3.3. Finally, we apply Lemma 3.4 to confirm that such a limit constitutes a symmetric PBE.

With additive separable buyer valuation, we have shown that a symmetric PBE exists for each finite market, and in any equilibrium a seller with higher quality posts a weakly higher reserve price. However, drawing further information about an equilibrium seems difficult given the complex strategic interaction. Therefore, in the next section, we study "limit equilibria" in large market following the approach of Peters and Severinov (1997) to achieve qualitative understanding about equilibria in large finite markets.

[11] We thank John Duggan for pointing out this fact to us.

4 Large Market

In this section, we consider a directed search environment which is the limit of our finite-market model as the number of players on both sides of the market increases at a constant ratio $k > 0$. We are particularly interested in efficiency and informativeness of "limit equilibria", and thus we mainly focus on (fully) separating seller behavior. Recall that we have established the following facts: (1) each type of buyer participates with uniform probabilities over auctions whose induced minimum types are low enough; (2) signaling high quality requires sufficient sacrifice of trade opportunities; (3) in any symmetric PBE, a lower-quality seller has a lower induced minimum type. These facts have an interesting implication: when the market is large, the demand side could be extremely thick at the lower end, which may locally form an obstacle to quality separation. In the large-market environment, we show that there could be no separating limit equilibrium in which the lowest-quality seller posts a reserve price equal to his opportunity cost.

4.1 Large-market model

Instead of building up a large-market model with full generality, we adopt a formulation tailored to analysis of separating limit equilibrium. Let $\sigma : \Theta \to S$ and $\mu : \Theta \to \Theta$ be a pair of continuous functions. There are $\underline{t} < \bar{t}$ in Θ such that (1) $\sigma([0, \underline{t}]) = \{\underline{s}\}$, $\sigma([\bar{t}, 1]) = \{\sigma(\bar{t})\}$, and σ is strictly increasing on $[\underline{t}, \bar{t}]$; (2) $\mu(\theta) = \theta$ on $[0, \underline{t})$, $\mu(\theta) \leq \theta$ is strictly increasing on $[\underline{t}, 1]$. σ and μ describe the matching technology of the market as follows: buyers with type below $\underline{t}$ are essentially excluded by all sellers; a seller targeting buyers of type $[t, 1]$ posts a reserve price of $\alpha(\mu(t)) + \beta(\sigma(t))$; a buyer of type t is supposed to visit sellers of types $[\underline{s}, \sigma(t)]$ with uniform probabilities, and when visiting a seller of type s, she bids $\alpha(t) + \beta(s)$. Such a matching technology captures the important features of finite markets with quality separation: there are monotonic links between "quality (s) – reserve price (r) – minimum type (m)– cutoff participation type (t) – sacrifice of trade opportunities (q)", and each buyer type visits uniformly sellers with low enough quality.

Each σ then determines the first and the second order statistics of an auction targeting $[t, 1]$:[12]

[12]Here we provide the intuition of the formulas. For formal discussions, we refer to Kim and Kircher (2015), Hernando-Veciana (2005) and Peters and Severinov (1997). In a finite market, the probability of seller j to attract a buyer is $p_j = \sum_{j' \geq j} \frac{F(t_{j'+1}) - F(t_{j'})}{j'}$. Since the number of buyers in each auction follows a multinomial distribution with probabilities p_j, as n tends to infinity the number of buyers that visit

$$F^1_{\sigma,t}(x) = e^{-k\int_{\max\{x,t\}}^1 \frac{f(\tilde{t})d\tilde{t}}{G(\sigma(\tilde{t}))}}$$

$$F^2_{\sigma,t}(x) = \left(1 + k\int_{\max\{x,t\}}^1 \frac{f(\tilde{t})d\tilde{t}}{G(\sigma(\tilde{t}))}\right) e^{-k\int_{\max\{x,t\}}^1 \frac{f(\tilde{t})d\tilde{t}}{G(\sigma(\tilde{t}))}}.$$

And the payoff to a type s seller targeting t is

$$\begin{aligned}
u_s(t) &= c(s)F^1_{\sigma,t}(t) && \text{(no trade)} \\
&\quad + (\alpha(\mu(t)) + \beta(\sigma(t))(F^2_{\sigma,t}(t) - F^1_{\sigma,t}(t)) && \text{(trade without buyer competition)} \\
&\quad + \int_t^1 (\alpha(x) + \beta(\sigma(t)))dF^2_{\sigma,t}(x), && \text{(trade with buyer competition)}
\end{aligned}$$

while the payoff to a type θ buyer who visits an auction targeting $t \leq \theta$ is

$$\begin{aligned}
v_\theta(t) &= (\alpha(\theta) - \alpha(\mu(t)))F^1_{\sigma,t}(t) && \text{(trade without buyer competition)} \\
&\quad + \int_t^\theta (\alpha(\theta) - \alpha(x))dF^1_{\sigma,t}(x). && \text{(trade with buyer competition)}
\end{aligned}$$

When there is no confusion about dependence on (σ, t), the i-th order statistic is written as F^i. Also let $f^i(x) = \frac{dF^i(x)}{dx}$ for $x \neq t$ and $f^i(t) = \lim_{x \to t^+} \frac{F^i(x) - F^i(t)}{x - t}$.

A *separating limit equilibrium* is defined as a pair (σ^*, μ^*) such that

$$\begin{aligned}
(\sigma^*)^{-1}(s) &\in \arg\max_{t \in \Theta} u_s(t), \forall s \in \text{int}S && \text{(incentive compatibility)} \\
\bar{t}^* &\in \arg\max_{t \in \Theta} u_{\sigma(\bar{t}^*)}(t) && \text{(no overpricing)} \\
\underline{t}^* &\in \arg\max_{t \in \Theta} u_{\underline{s}}(t) && \text{(no undercutting)} \\
[\underline{t}^*, \theta] &\subset \arg\max_{t \in \Theta} v_\theta(t), \forall \theta \geq \underline{t}^*. && \text{(optimal search)}
\end{aligned}$$

4.2 Non-existence of separating limit equilibrium

Now we proceed to the analysis of separating limit equilibria. For technical reasons, we assume that g, f, α, β and c are $\mathcal{C}^\infty$ and Lipschitz. Throughout this subsection, we also assume that $\alpha^{-1}(c(\underline{s}) - \beta(\underline{s})) \geq 0$, i.e., the lowest-quality seller still excludes some types

each auction follows a Poisson distribution with rate $\lambda_j = \lim_{N\to\infty} kNp_j = \lim_{N\to\infty} k\sum_{j'=j}^N \frac{F(t_{j'+1}) - F(t_{j'})}{j'/N} = k\int_{t_j}^1 \frac{f(\tilde{t})d\tilde{t}}{G(\sigma(\tilde{t}))}$.

of buyers when he posts a reserve price equal to his opportunity cost.[13] We first provide an ODE-system characterization of separating limit equilibria. Then we show that the relevant initial value problem has a solution which violates the strict monotonicity of a separating limit equilibrium. Finally we study conditions under which this solution is unique and thus there is no separating equilibrium.

If (σ, μ) is a separating limit equilibrium, then they are C^1 on $(\underline{t}, \bar{t})$ (see also Mailath (1987) and Mailath (1988)). The following lemma presents the first-order conditions for (σ, μ) in equilibrium, and it also implies that σ and μ are C^∞ on $(\underline{t}, \bar{t})$.

Lemma 4.1. If (σ, μ) is a separating limit equilibrium, then on $(\underline{t}, \bar{t})$

$$\begin{aligned}\frac{d\sigma}{dt} &= \frac{\alpha(\mu(t)) + \beta(\sigma(t)) - c(\sigma(t))}{\beta'(\sigma(t))} \frac{f^1(t)}{1 - F^1(t)} \\ \frac{d\mu}{dt} &= \frac{f^1(t)}{F^1(t)} \frac{\alpha(t) - \alpha(\mu(t))}{\alpha'(\mu(t))}.\end{aligned}$$

Proof (1). By integration by parts, for $t \leq \theta$,

$$\begin{aligned}v_\theta(t) &= (\alpha(\theta) - \alpha(\mu(t)))F^1(t) + \int_t^\theta (\alpha(\theta) - \alpha(x))dF^1(x) \\ &= (\alpha(t) - \alpha(\mu(t)))F^1(t) + \int_t^\theta F^1(x)d\alpha(x)\end{aligned}$$

and thus

$$\frac{dv_\theta}{dt} = -\alpha'(\mu(t))\frac{d\mu}{dt}F^1(t) + (\alpha(t) - \alpha(\mu(t)))f^1(t).$$

Since a type θ buyer is supposed to be indifferent between $(\underline{t}, \theta]$, we have

$$\frac{d\mu}{dt} = \frac{f^1(t)}{F^1(t)} \frac{\alpha(t) - \alpha(\mu(t))}{\alpha'(\mu(t))}.$$

[13] The other case would be discussed in the end of this section.

(2). Note that

$$
\begin{aligned}
\frac{du_s}{dt} =& c(s)f^1(t) + \left(\alpha'(\mu(t))\frac{d\mu}{dt} + \beta'(\sigma(t))\frac{d\sigma}{dt}\right)(F^2(t) - F^1(t)) \\
&+ ((\alpha(\mu(t)) + \beta(\sigma(t)))(f^2(t) - f^1(t)) \\
&- (\alpha(t) + \beta(\sigma(t)))f^2(t) + \beta'(\sigma(t))\frac{d\sigma}{dt}(1 - F^2(t))) \\
=& -\big(\alpha(\mu(t)) + \beta(\sigma(t)) - c(s)\big) f^1(t) + \beta'(\sigma(t))\frac{d\sigma}{dt}(1 - F^1(t)) \\
&+ \alpha'(\mu(t))\frac{d\mu}{dt}(F^2(t) - F^1(t)) - (\alpha(t) - \alpha(\mu(t)))f^2(t).
\end{aligned}
$$

By (1) and the fact that $f^1(t)\frac{F^2(t)-F^1(t)}{F^1(t)} = f^2(t)$,

$$
\alpha'(\mu(t))\frac{d\mu}{dt}(F^2(t) - F^1(t)) - (\alpha(t) - \alpha(\mu(t)))f^2(t) = 0
$$

Then we obtain a simple expression of $\frac{du_s}{dt}$ after taking into account of buyers' behavior

$$
\frac{du_s}{dt} = -\big(\alpha(\mu(t)) + \beta(\sigma(t)) - c(s)\big) f^1(t) + \beta'(\sigma(t))\frac{d\sigma}{dt}(1 - F^1(t)), \tag{$*_s$}
$$

and incentive compatibility implies

$$
\frac{d\sigma}{dt} = \frac{\alpha(\mu(t)) + \beta(\sigma(t)) - c(\sigma(t))}{\beta'(\sigma(t))}\frac{f^1(t)}{1 - F^1(t)}.
$$

From ($*_s$), we know that for each $s > \underline{s}$, the posted reserve price is strictly higher than his opportunity cost.[14] Also if the action of a seller has no signaling effect, then it is optimal for him to post a reserve price equal to his opportunity cost.[15] Therefore, there is an immediate distortion from adverse selection compared to the environment with quality disclosure. And we will show that the interaction between adverse selection and search friction generate further distortions.

Let $q(t) = F^1(t)$, then $\frac{dq}{dt} = k\frac{f(t)}{G(\sigma(t))}q(t)$. Then the system in Lemma 4.1 can be written

[14] One can show that if there is $\hat{t} \in (\underline{t}, \bar{t})$ such that $\alpha(\mu(\hat{t})) + \beta(\sigma(\hat{t})) = c(\sigma(\hat{t}))$, then $\frac{d\sigma}{dt} < 0$ at some $t' \in (\underline{t}, \hat{t})$, which contradicts that σ constitutes a separating limit equilibrium.

[15] It is easy to see that when qualities are disclosed, $\frac{du_s}{dt} = -\big(\alpha(\mu(t)) + \beta(s) - c(s)\big) f^1(t)$.

as

$$\frac{d\sigma}{dt} = \frac{\alpha(\mu(t)) + \beta(\sigma(t)) - c(\sigma(t))}{\beta'(\sigma(t))} \frac{1}{1-q(t)} \frac{dq}{dt}$$
$$\frac{d\mu}{dt} = \frac{\alpha(t) - \alpha(\mu(t))}{\alpha'(\mu(t))} \frac{1}{q(t)} \frac{dq}{dt}$$
$$\frac{dq}{dt} = k\frac{f(t)}{G(\sigma(t))} q(t).$$

Since $\frac{dq}{dt} > 0$ on $(\underline{t}, \bar{t})$, it is valid to consider q as the time variable and we obtain the following transformed ODE system:[16]

$$\frac{d\sigma}{dq} = \frac{1}{1-q} \cdot \frac{\alpha(\mu) + \beta(\sigma) - c(\sigma)}{\beta'(\sigma)}$$
$$\frac{d\mu}{dq} = \frac{1}{q} \cdot \frac{\alpha(t) - \alpha(\mu)}{\alpha'(\mu)}$$
$$\frac{dt}{dq} = \frac{1}{q} \cdot \frac{G(\sigma)}{kf(t)}. \tag{$*$}$$

In the following analysis, σ, μ and t are regarded as functions of q. Note that if they constitute a separating limit equilibrium, they are strictly increasing. As for the initial value of the system, set $\sigma_0 = \underline{s}$. Also let $t_* = \alpha^{-1}(c(\underline{s}) - \beta(\underline{s}))$ be the induced minimum type if a type $\underline{s}$ seller reveals his quality and posts a reserve price equal to his opportunity cost. t_* seems to be a natural focus for the choice of μ_0 and t_0 as it is the optimal choice for $\underline{s}$ when qualities are disclosed. Moreover, we show in the next lemma that t_* is closely related to the no-undercutting criterion: if the lowest-quality seller can not trade with probability one at a reserve price strictly higher than his opportunity cost, he prefers to undercut the market. And thus it guarantees that the corresponding equilibrium behavior is robust to the introduction of a small positive measure of type $\underline{s}$ sellers who always match $\underline{t}$.

Lemma 4.2. In any separating limit equilibrium, $F^1(\underline{t}) > 0$ implies $\alpha(\underline{t}) + \beta(\underline{s}) = c(\underline{s})$.

Proof The payoff for a type-$\underline{s}$ seller to undercut the market,[17] i.e., to target a buyer type

[16]Note that the two ODE systems are well-defined only for $(q, \sigma, \mu, t) \in (0,1) \times (\underline{s}, \bar{s}] \times (\underline{t}, 1] \times (\underline{t}, 1]$. Our analysis involves cases in which the systems are not well-defined at the initial values. In such cases, a solution to an initial value problem is understood as a function which is continuous at the initial value and satisfies the ODE system when it is well-defined. Then our transformation is valid in the sense of preserving solutions to the original problem.

[17]See also Section 4 of Peters and Severinov (1997).

below $\underline{t}$, is

$$u_{\underline{s}}(\underline{t}^-) = (\alpha(\underline{t}) + \beta(\underline{s}))F^2(\underline{t}) + \int_{\underline{t}^+}^1 (\alpha(x) + \beta(\underline{s}))dF^2(x)$$

No undercutting requires $u_{\underline{s}}(\underline{t}) \geq u_{\underline{s}}(\underline{t}^-)$ and thus

$$[c(\underline{s}) - (\alpha(\underline{t}) + \beta(\underline{s}))] \cdot F^1(\underline{t}) \geq 0.$$

Therefore, $\alpha(\underline{t}) + \beta(\underline{s}) \leq c(\underline{s})$; and by incentive compatibility $\alpha(\underline{t}) + \beta(\underline{s}) = c(\underline{s})$.

Consider $q_0 > 0$. Lemma 4.2 implies that $\mu_0 = t_0 = t_*$. It is easy to see that the constant function $(\sigma(q), \mu(q), t(q)) = (\sigma_0, \mu_0, t_0)$ is a solution to the system $(*)$ with initial value $(q_0, \underline{s}, t_*, t_*)$. This solution is also unique as the initial value problem is Lipschitz on $[q_0, 1-\varepsilon]$ with $\varepsilon > 0$ arbitrarily small. Since the constant function violates strict monotonicity of a separating limit equilibrium, it is impossible that $q_0 > 0$ in equilibrium. Therefore, in search of a separating limit equilibrium, we focus on the solution to the system $(*)$ with initial value $(0, \underline{s}, t_*, t_*)$. We refer to this initial value problem as $(*_0)$.

When $q_0 = 0$, the constant function is a solution to $(*_0)$ but uniqueness is not straightforward as the problem is no longer Lipschitz. Our second main result establishes that if the buyer-seller ratio k is sufficiently large, then the constant function is the unique solution to $(*_0)$, and thus there is no separating limit equilibrium. Without the condition on buyer-seller ratio, the constant function remains unique in the class of solutions satisfying the following regularity condition: we say that $x(q) : [0,1] \to S \times \Theta \times \Theta$ is *regular*, if for each $n \in \mathbb{N}$, the limit of the n-th order derivative of $x(q)$ as $q \to 0$ exists (either finite or infinite).

Theorem 2. (1) If $k > \frac{g(\underline{s})}{2f(t_*)}$, then there is no separating limit equilibrium. (2) If $k \leq \frac{g(\underline{s})}{2f(t_*)}$, there is no separating limit equilibrium such that $(\sigma(q), \mu(q), t(q))$ is *regular*.

The crux of the proof is a modification of the technique in Gard (1978) who establishes a generalized uniqueness criterion for initial value problems of ODE systems. We show that if $\frac{g(\underline{s})}{kf(t_*)} < M \in \mathbb{N}$ and for each $n \leq M$, the limit of the n-th order derivative of $x(q)$ as $q \to 0$ exists, then there is at most one solution $x(q)$ to the problem $(*_0)$. The strict monotonicity of a separating limit equilibrium allows us to show that for any solution to the problem $(*_0)$, such limits exist for $n \in \{1, 2\}$. Therefore, if $\frac{g(\underline{s})}{kf(t_*)} < 2$ or if we restrict attention to *regular* solutions, the constant function is the unique solution to $(*_0)$. Details of the proof are in Appendix C.

In the case where both F and G represent the uniform distribution on the unit interval, the cutoff value for k is $\frac{1}{2}$: if there is more buyers than sellers in the market, there is no separating limit equilibrium; in fact, as long as the number of buyers are more than half of the number of sellers, there is no separating limit equilibrium. When k is sufficiently small, a separating limit equilibrium may exist but the corresponding mapping $(\sigma(q), \mu(q), t(q))$ cannot be regular in the sense we have defined.

Now we focus on $k > \frac{g(\underline{s})}{2f(t_*)}$. Note that if sellers with quality in an interval separate in a limit equilibrium, then the ODE system ($*$) is satisfied on the corresponding set of variables. And the proof of Theorem 2 actually indicates that there is no limit equilibrium in which the lower end of the supply side fully reveals quality and the lowest-quality seller posts a reserve price equal to his opportunity cost. Therefore, in a limit equilibrium, either there is $\hat{s} > \underline{s}$ such that sellers with quality in $[\underline{s}, \hat{s}]$ post the same reserve price, or the seller behavior is separating but the lowest-quality seller posts a reserve price strictly higher than his opportunity cost.[18] In either case, the interaction between adverse selection and search friction entails further distortions at the lower end of the market and leads to extra efficiency loss. These distortions carry over to large finite markets. Suppose a sequence of finite-market equilibria converge to a limit equilibrium in which sellers with qualities in $[\underline{s}, \hat{s}]$ post the same reserve price. By Lemma 3.4, if $s_1, s_2 \in [\underline{s}, \hat{s}]$ separate along the sequence, then they separate in the limit. Therefore, in sufficiently large finite markets, s_1, s_2 also post the same reserve price. In the other case, suppose a sequence of finite-market equilibria converge to a limit equilibrium in which the lowest-quality seller posts a reserve price $\underline{r} > c(\underline{s})$. It is possible that at the lower end of a finite market, posted reserve prices are close to opportunity costs. However, such behaviors vanish when the market becomes large. For each $\epsilon > 0$, a seller of type $\underline{s} + \epsilon$ will post a reserve price strictly higher than $\underline{r}$ in sufficiently large finite markets. In both cases, distortions are present in large finite markets.

Remark 4.3 We would like to leave these interesting and challenging tasks for future research:

(1) When $k > \frac{g(\underline{s})}{2f(t_*)}$, given what we learn from Theorem 2, a natural question would be how model primitives affect the distortions. The ODE system ($*$) provides a tool to analyze the class of limit equilibria parametrized by $\hat{s} \in S$ such that $[\underline{s}, \hat{s}]$ pool at the same reserve price, while $[\hat{s}, \overline{s}]$ separate. For each $\hat{s}$ and a guess of the "initial" probability

[18] As for the second case, one can notice that Lemma 4.2 and Theorem 2 leave open the possibility of a separating limit equilibrium with $F^1(\underline{t}) = 0$ and $\underline{t} > t_*$, but as we have discussed, such equilibrium behavior fails a naive robustness check.

of no trade, minimum type and cutoff participation type, the corresponding solution to the initial value problem generates a candidate for limit equilibria. As verification, the probability of no trade for $\hat{s}$ computed at the candidate should coincide with the guess; meanwhile, the candidate should also satisfy $t(1) = 1$, i.e., the cutoff participation type is the highest buyer type when the associated probability of no trade is 1. This "guess and verify" approach is equivalent to an analysis of the ODE system ($*$) with multidimensional boundary value. When $k \leq \frac{g(\underline{s})}{2f(t_*)}$, the technique we adopt can not address whether there is a separating limit equilibrium that behaves irregularly at $q = 0$.

(2) Also recall that we have assumed $\alpha^{-1}(c(\underline{s}) - \beta(\underline{s})) \geq 0$ in this subsection.[19] This assumption implies $\mu_0 = t_0$ when the initial value for the ODE system ($*$) is set. When $\alpha^{-1}(c(\underline{s}) - \beta(\underline{s})) < 0$, we would have $\mu_0 < t_0$ accordingly. For $q_0 > 0$, an ODE system for separating limit equilibrium with μ as the time variable is well-behaved. But the main difficulty lies in the analysis for $q_0 = 0$. Existence and uniqueness of a solution to the corresponding initial value problem are unclear.

[19] This is a "no gap" assumption that precludes certainty of gain from trade. However, there is no logical link between "no gap" and non-existence of separating equilibrium in general. Cai, Riley and Ye (2007) shows the existence of a separating equilibrium in "no gap" cases when there is a single seller.

5 Conclusion

In this paper, we study competing auctions with informed sellers. We show existence of a symmetric PBE in any finite market. And in any PBE, higher quality is signaled through higher reserve price and at the expense of trade opportunities. By analyzing the large-market model, we also find that the interaction of adverse selection and search friction can lead to distortion at the lower end of the market. When the buyer-seller ratio is sufficiently large or if a regularity condition is imposed, there is no separating limit equilibrium in which the lowest-quality seller posts a reserve price equal to his opportunity cost.

Our main results rely on the assumption that a buyer's valuation of object is additively separable in her own type and the quality of the object. For other forms of valuation, existence of PBE in finite markets when the space of reserve price is a continuum remains unknown; and in large markets, we conjecture that a distortion at the lower end would persist and there could be interesting behavior at the higher end when there are quality searching types.[20] Note that in our model, each buyer makes her participation/search decision after she observes her own type. This is a natural assumption in some contexts. But for other contexts including experience goods, a study of the alternative model in which buyers search before they observe their own type is also important.

[20] See Appendix D.

A Comparative statics in minimum type

Recall that

$$q_j = 1 - \sum_{j' \geq j} \frac{F(t_{j'+1}) - F(t_{j'})}{j'}.$$

in which $(t_j)_{j \in J}$ characterizes buyers' participation/search behavior in response to $(m_j)_{j \in J}$, and we set $m_0 = 0$ and $m_{|J|+1} = 1$.

Proposition 3.2. Suppose $m_j \in (m_{j-1}, m_{j+1})$ and $q_j < 1$, then $\frac{\partial q_j}{\partial m_j} > 0$.

Proof We focus on the case in which $m_1 < m_2 < ... < m_J$ with induced $t_J^* < 1$; other cases can be shown with similar logic. From Hernando-Veciana (2005), we know that

$$m_1 = t_1^* < t_2^* < ... < t_J^*$$

and for each $j \geq 2$,

$$m_j < t_j^*.$$

For each $j \geq 2$, let

$$\varphi_j = \alpha(m_{j-1}) \left(\frac{q_{j-1}}{q_j} \right)^{kJ-1} + \int_{t_{j-1}}^{t_j} \alpha(x) d \left(\frac{\tilde{q}_{j-1}(x)}{q_j} \right)^{kJ-1} - \alpha(m_j)$$

in which

$$\tilde{q}_{j-1}(x) = 1 - \left(\frac{F(t_j) - F(x)}{j-1} + \frac{F(t_{j+1}) - F(t_j)}{j} + ... \right)$$

Note that φ_j is essentially the function Λ_j in Hernando-Veciana (2005), and in the case under consideration, (1) $\frac{\partial \varphi_j}{\partial t_{j-1}} < 0$; (2) for $\tilde{j} \geq j$, $\frac{\partial \varphi_j}{\partial t_{\tilde{j}}} > 0$; (3) $\frac{\partial \varphi_j}{\partial m_{j-1}} > 0$ and $\frac{\partial \varphi_j}{\partial m_j} < 0$; (4) φ_j is independent of $m_{\tilde{j}}$ if $\tilde{j} \notin \{j-1, j\}$, and it is also independent of $t_{\tilde{j}}$ with $\tilde{j} \leq j-2$. Moreover, the system $(\varphi_2, ..., \varphi_J) = \mathbf{0}$ parametrized by $(m_1, ..., m_J)$ has a unique solution $(t_2^*, ..., t_J^*)$.

Fix $j \in \{2, ..., J-1\}$. we want to show that (1) $\frac{\partial t_j^*}{\partial m_j} > 0$ and (2) for $\tilde{j} > j$, $\frac{\partial t_{\tilde{j}}^*}{\partial m_j} < 0$, which then imply

$$\frac{\partial q_j}{\partial m_j} > 0,$$

i.e., the probability of no trade for seller j is strictly increasing in his induced minimum type taking into account the equilibrium behavior of buyers. The idea of the following argument is to establish the sufficiency of analyzing φ_j and φ_{j+1}.

First, we claim that for each pair of $j_1, j_2 > j$,

$$\operatorname{sgn}\left(\frac{\partial t^*_{j_1}}{\partial m_j}\right) = \operatorname{sgn}\left(\frac{\partial t^*_{j_2}}{\partial m_j}\right).$$

For each $\tilde{j} \in \{j+1, ..., J-1\}$, applying $\frac{\partial}{\partial m_j}$ to $\varphi_{\tilde{j}+1}$ yields

$$\frac{\partial \varphi_{\tilde{j}+1}}{\partial t_{\tilde{j}}} \frac{\partial t^*_{\tilde{j}}}{\partial m_j} + \sum_{j' \geq \tilde{j}+1} \frac{\partial \varphi_{\tilde{j}+1}}{\partial t_{j'}} \frac{\partial t^*_{j'}}{\partial m_j} = 0.$$

Since $\frac{\partial \varphi_{\tilde{j}+1}}{\partial t_{\tilde{j}}} < 0$ and $\frac{\partial \varphi_{\tilde{j}+1}}{\partial t_{j'}} > 0$, we have $\operatorname{sgn}\left(\frac{\partial t^*_{j_1}}{\partial m_j}\right) = \operatorname{sgn}\left(\frac{\partial t^*_{j_2}}{\partial m_j}\right)$ and the claim follows immediately from induction.

Second, we claim that to show $\frac{\partial t^*_j}{\partial m_j} > 0$ and $\frac{\partial t^*_{\tilde{j}}}{\partial m_j} < 0$ for $\tilde{j} > j$, it suffices to show that

$$\frac{\partial \varphi_j}{\partial t_j} \frac{\partial \varphi_{j+1}}{\partial m_j} - \frac{\partial \varphi_{j+1}}{\partial t_j} \frac{\partial \varphi_j}{\partial m_j} > 0$$

if evaluated at $\mathbf{t}^*$. To see this, apply $\frac{\partial}{\partial m_j}$ to $(\varphi_2, ..., \varphi_{j+1})$. For $j = 2$, notice that

$$\frac{\partial \varphi_2}{\partial t_2} \frac{\partial t^*_2}{\partial m_2} + \sum_{j' \geq 3} \frac{\partial \varphi_2}{\partial t_{j'}} \frac{\partial t^*_{j'}}{\partial m_2} + \frac{\partial \varphi_2}{\partial m_2} = 0$$
$$\frac{\partial \varphi_3}{\partial t_2} \frac{\partial t^*_2}{\partial m_2} + \sum_{j' \geq 3} \frac{\partial \varphi_3}{\partial t_{j'}} \frac{\partial t^*_{j'}}{\partial m_2} + \frac{\partial \varphi_3}{\partial m_2} = 0.$$

In the first equation, since $\frac{\partial \varphi_2}{\partial t_2} > 0$, $\frac{\partial \varphi_2}{\partial t_{j'}} > 0$ and $\frac{\partial \varphi_2}{\partial m_2} < 0$, it is easy to see that if $\frac{\partial t^*_{j'}}{\partial m_2} < 0$ for each $j' \geq 3$, then $\frac{\partial t^*_2}{\partial m_2} > 0$. Eliminating $\frac{\partial t^*_2}{\partial m_2}$ and we have

$$\sum_{j' \geq 3} \left(\frac{\partial \varphi_2}{\partial t_2} \frac{\partial \varphi_3}{\partial t_{j'}} - \frac{\partial \varphi_3}{\partial t_2} \frac{\partial \varphi_2}{\partial t_{j'}} \right) \frac{\partial t^*_{j'}}{\partial m_2} = -\left(\frac{\partial \varphi_2}{\partial t_2} \frac{\partial \varphi_3}{\partial m_2} - \frac{\partial \varphi_3}{\partial t_2} \frac{\partial \varphi_2}{\partial m_2} \right).$$

Since $\frac{\partial \varphi_2}{\partial t_2} \frac{\partial \varphi_3}{\partial t_{j'}} - \frac{\partial \varphi_3}{\partial t_2} \frac{\partial \varphi_2}{\partial t_{j'}} > 0$, $\frac{\partial \varphi_2}{\partial t_2} \frac{\partial \varphi_3}{\partial m_2} - \frac{\partial \varphi_3}{\partial t_2} \frac{\partial \varphi_2}{\partial m_2} > 0$ implies $\frac{\partial t^*_{j'}}{\partial m_2} < 0$ and thus $\frac{\partial t^*_2}{\partial m_2} > 0$. This argument can be extended to $j > 2$ by repeated elimination of $\frac{\partial t^*_{\tilde{j}}}{\partial m_j}$ for $\tilde{j} < j$. To be more precise, after elimination of $\frac{\partial t^*_2}{\partial m_j}, ..., \frac{\partial t^*_{j-2}}{\partial m_j}$, there are functions $\left\{A_{\tilde{j}}\right\}_{\tilde{j} \geq j-1}$ such that for each

$\tilde{j}$, $A_{\tilde{j}} > 0$, and $A_{j-1}\frac{\partial t^*_{j-1}}{\partial m_j} + \sum_{j' \geq j} A_{j'}\frac{\partial t^*_{j'}}{\partial m_j} = 0$. Now consider

$$A_{j-1}\frac{\partial t^*_{j-1}}{\partial m_j} + \sum_{j' \geq j} A_{j'}\frac{\partial t^*_{j'}}{\partial m_j} = 0$$
$$\frac{\partial \varphi_j}{\partial t_{j-1}}\frac{\partial t^*_{j-1}}{\partial m_j} + \sum_{j' \geq j}\frac{\partial \varphi_j}{\partial t_{j'}}\frac{\partial t^*_{j'}}{\partial m_j} + \frac{\partial \varphi_j}{\partial m_j} = 0$$

and eliminate $\frac{\partial t^*_{j-1}}{\partial m_j}$. We have

$$\left(A_{j-1}\frac{\partial \varphi_j}{\partial t_j} - \frac{\partial \varphi_j}{\partial t_{j-1}}A_j\right)\frac{\partial t^*_j}{\partial m_j} + \sum_{j' \geq j+1}\left(A_{j-1}\frac{\partial \varphi_j}{\partial t_{j'}} - \frac{\partial \varphi_j}{\partial t_{j-1}}A_{j'}\right)\frac{\partial t^*_{j'}}{\partial m_j} + A_{j-1}\frac{\partial \varphi_j}{\partial m_j} = 0$$

and thus if $\frac{\partial t^*_{j'}}{\partial m_j} < 0$ for each $j' \geq j+1$, then $\frac{\partial t^*_j}{\partial m_j} > 0$. Then eliminate $\frac{\partial t^*_j}{\partial m_j}$ according to

$$\left(A_{j-1}\frac{\partial \varphi_j}{\partial t_j} - \frac{\partial \varphi_j}{\partial t_{j-1}}A_j\right)\frac{\partial t^*_j}{\partial m_j} + \sum_{j' \geq j+1}\left(A_{j-1}\frac{\partial \varphi_j}{\partial t_{j'}} - \frac{\partial \varphi_j}{\partial t_{j-1}}A_{j'}\right)\frac{\partial t^*_{j'}}{\partial m_j} + A_{j-1}\frac{\partial \varphi_j}{\partial m_j} = 0$$
$$\frac{\partial \varphi_{j+1}}{\partial t_j}\frac{\partial t^*_j}{\partial m_j} + \sum_{j' \geq j+1}\frac{\partial \varphi_{j+1}}{\partial t_{j'}}\frac{\partial t^*_{j'}}{\partial m_j} + \frac{\partial \varphi_{j+1}}{\partial m_j} = 0,$$

and we have if $(A_{j-1}\frac{\partial \varphi_j}{\partial t_j} - \frac{\partial \varphi_j}{\partial t_{j-1}}A_j)\frac{\partial \varphi_{j+1}}{\partial m_j} - A_{j-1}\frac{\partial \varphi_{j+1}}{\partial t_j}\frac{\partial \varphi_j}{\partial m_j} > 0$, then $\frac{\partial t^*_{j'}}{\partial m_j} < 0$. Note that

$$\begin{aligned}&\left(A_{j-1}\frac{\partial \varphi_j}{\partial t_j} - \frac{\partial \varphi_j}{\partial t_{j-1}}A_j\right)\frac{\partial \varphi_{j+1}}{\partial m_j} - A_{j-1}\frac{\partial \varphi_{j+1}}{\partial t_j}\frac{\partial \varphi_j}{\partial m_j}\\ =&A_{j-1}\left(\frac{\partial \varphi_j}{\partial t_j}\frac{\partial \varphi_{j+1}}{\partial m_j} - \frac{\partial \varphi_{j+1}}{\partial t_j}\frac{\partial \varphi_j}{\partial m_j}\right) - A_j\frac{\partial \varphi_j}{\partial t_{j-1}}\frac{\partial \varphi_{j+1}}{\partial m_j}\end{aligned}$$

and
$$A_j\frac{\partial \varphi_j}{\partial t_{j-1}}\frac{\partial \varphi_{j+1}}{\partial m_j} < 0.$$

Therefore, $\frac{\partial \varphi_j}{\partial t_j}\frac{\partial \varphi_{j+1}}{\partial m_j} - \frac{\partial \varphi_{j+1}}{\partial t_j}\frac{\partial \varphi_j}{\partial m_j} > 0$ implies $\frac{\partial t^*_{j'}}{\partial m_j} < 0$ and thus $\frac{\partial t^*_j}{\partial m_j} > 0$.

Finally, we show that indeed $\frac{\partial \varphi_j}{\partial t_j}\frac{\partial \varphi_{j+1}}{\partial m_j} > \frac{\partial \varphi_{j+1}}{\partial t_j}\frac{\partial \varphi_j}{\partial m_j}$ if evaluated at $\mathbf{t}^*$. Recall that

$$\varphi_j = \alpha(m_{j-1})(\frac{q_{j-1}}{q_j})^{kJ-1} + \int_{t_{j-1}}^{t_j} \alpha(x) d(\frac{\tilde{q}_{j-1}(x)}{q_j})^{kJ-1} - \alpha(m_j)$$
$$\varphi_{j+1} = \alpha(m_j)(\frac{q_j}{q_{j+1}})^{kJ-1} + \int_{t_j}^{t_{j+1}} \alpha(x) d(\frac{\tilde{q}_j(x)}{q_{j+1}})^{kJ-1} - \alpha(m_{j+1}).$$

Then

$$\frac{\partial \varphi_j}{\partial t_j}\frac{\partial \varphi_{j+1}}{\partial m_j} - \frac{\partial \varphi_{j+1}}{\partial t_j}\frac{\partial \varphi_j}{\partial m_j} = \alpha'(m_j)\left[\frac{\partial \varphi_j}{\partial t_j}(\frac{q_j}{q_{j+1}})^{kJ-1} + \frac{\partial \varphi_{j+1}}{\partial t_j}\right]$$

and thus it is equivalent to show

$$\frac{\partial \varphi_j}{\partial t_j}\left(\frac{q_j}{q_{j+1}}\right)^{kJ-1} + \frac{\partial \varphi_{j+1}}{\partial t_j} > 0.$$

Since q_{j+1} is independent of t_j, we have

$$\begin{aligned}\frac{\partial \varphi_{j+1}}{\partial t_j} &= \alpha(m_j)(kJ-1)\frac{(q_j)^{kJ-2}}{(q_{j+1})^{kJ-1}}\frac{\partial q_j}{\partial t_j} \\ &\quad - \alpha(t_j)(kJ-1)\frac{(q_j)^{kJ-2}}{(q_{j+1})^{kJ-1}}\left(\frac{\partial \tilde{q}_j}{\partial x}|_{x=t_j}\right) \\ &= (\alpha(m_j) - \alpha(t_j))(\frac{q_j}{q_{j+1}})^{kJ-1}\frac{(kJ-1)f(t_j)}{jq_j}\end{aligned}$$

and it is equivalent to show

$$\frac{\partial \varphi_j}{\partial t_j}\frac{jq_j}{(kJ-1)f(t_j)} > \alpha(t_j) - \alpha(m_j).$$

By integration by parts,

$$\begin{aligned}\varphi_j &= \alpha(m_{j-1})\left(\frac{q_{j-1}}{q_j}\right)^{kJ-1} + \int_{t_{j-1}}^{t_j} \alpha(x) d\left(\frac{\tilde{q}_{j-1}(x)}{q_j}\right)^{kJ-1} - \alpha(m_j) \\ &= -(\alpha(t_{j-1}) - \alpha(m_{j-1}))\left(\frac{q_{j-1}}{q_j}\right)^{kJ-1} \\ &\quad + (\alpha(t_j) - \alpha(m_j)) - \int_{t_{j-1}}^{t_j}\left(\frac{\tilde{q}_{j-1}(x)}{q_j}\right)^{kJ-1} d\alpha(x).\end{aligned}$$

An explicit derivation of $\frac{\partial \varphi_j}{\partial t_j}$ yields

$$\begin{aligned}\frac{\partial \varphi_j}{\partial t_j}\frac{jq_j}{(kJ-1)f(t_j)} &= (\alpha(t_{j-1})-\alpha(m_{j-1}))\left[\frac{1}{j-1}\left(\frac{q_{j-1}}{q_j}\right)^{kJ-2}+\left(\frac{q_{j-1}}{q_j}\right)^{kJ-1}\right] \\ &\quad + \int_{t_{j-1}}^{t_j}\left[\frac{1}{j-1}\left(\frac{\tilde{q}_{j-1}(x)}{q_j}\right)^{kJ-2}+\left(\frac{\tilde{q}_{j-1}(x)}{q_j}\right)^{kJ-1}\right]d\alpha(x) \\ &> (\alpha(t_{j-1})-\alpha(m_{j-1}))\left(\frac{q_{j-1}}{q_j}\right)^{kJ-1}+\int_{t_{j-1}}^{t_j}\left(\frac{\tilde{q}_{j-1}(x)}{q_j}\right)^{kJ-1}d\alpha(x).\end{aligned}$$

Since $\varphi_j = 0$ at $\mathbf{t}^*$, we have

$$\left(\frac{\partial \varphi_j}{\partial t_j}\frac{jq_j}{(kJ-1)f(t_j)}\right)|_{\mathbf{t}^*} > \alpha(t_j^*)-\alpha(m_j),$$

or equivalently $\frac{\partial \varphi_j}{\partial t_j}\frac{\partial \varphi_{j+1}}{\partial m_j} > \frac{\partial \varphi_{j+1}}{\partial t_j}\frac{\partial \varphi_j}{\partial m_j}$ if evaluated at $\mathbf{t}^*$, which completes the proof.

B Existence of symmetric PBE

First we introduce some notations. In this section, we denote the common prior over S as λ_0. Let d_H be a Hausdorff metric on $\mathbb{R}$. For $m,n \in \mathbb{N}$, let R_n be a finite subset of R with $d_H(R_n, R) < 2^{-n}$; also let S_m be a finite subset of S such that $d_H(S_m, S) < 2^{-m}$ and $\{\min S, \max S\} \subset S_m$. Then S_m can be enumerated as $\min S = s_{m,1} < s_{m,2} < \ldots < s_{m,|S_m|} = \max S$ of S. For $1 \le k \le |S_m|-1$, let $I_{m,k} = [s_{m,k}, s_{m,k+1})$; and $I = \left\{s_{m,|S_m|}\right\}$. Let Σ_m be the σ-algebra generated by $\{I_{m,k}\}$. Given n, a mixed strategy profile of a seller in $\Gamma(R_n, S)$ is regarded as a transition probability $\rho : (S, \mathcal{B}(S)) \to \Delta(R_n)$. Let $\mathcal{T}((S,\mathcal{B}(S)), R_n, \lambda_0)$, the space of transition probabilities from $(S, \mathcal{B}(S))$ to R, be equipped with the topology of weak convergence. For each m, $\mathcal{T}((S,\Sigma_m), R_n, \lambda_0)$ is equipped with the subspace topology inherited from $\mathcal{T}((S,\mathcal{B}(S)), R_n, \lambda_0)$. A belief updating scheme $\Lambda : R_n \to \Delta(S)$ is regarded as an element of $(\Delta(S))^{R_n}$ with the product weak* topology. (ρ, Λ) also generates a probability measure $\xi_{\rho,\Lambda}$ over $R \times \Delta(S)$. Let $\Delta(R \times \Delta(S))$ be equipped with the weak* topology. Note that the topological spaces above are compact and convex. Fix $B_0 \in \mathcal{B}(S)$, $\lambda_0 \cdot \mathbf{1}_{B_0}$ denotes the measure induced by

λ_0 truncated by B_0, i.e., $\forall B \in \mathcal{B}(S)$,

$$\left(\lambda_0 \cdot \mathbf{1}_{B_0}\right)(B) = \lambda_0\left(B \cap B_0\right).$$

When $\lambda_0(B_0) > 0$, $\frac{\lambda_0 \cdot \mathbf{1}_{B_0}}{\lambda_0(B_0)}$ s a probability measure; when $B_0 = \{s_0\}$, we regard $\frac{\lambda_0 \cdot \mathbf{1}_{B_0}}{\lambda_0(B_0)}$ as the Dirac measure δ_{s_0}.

Lemma 3.4. Suppose (ρ, Λ) satisfies incentive compatibility on supp (ρ), $r_1 > r_2$ are both on-path, $s_1 \in \text{supp}(\Lambda(r_1))$ and $s_2 \in \text{supp}(\Lambda(r_2))$. Then there is $D(s_1, s_2) > 0$ such that $r_1 - r_2 > D(s_1, s_2)$. Moreover, $D(s_1, s_2)$ is continuous, decreasing in s_1 and increasing in s_2.

Proof Let $I_1, I_2 \subset S$ be compact intervals. Consider the following problem:

$$\begin{aligned} \inf_{I_1, I_2} & \beta\left(\frac{\lambda_0 \cdot \mathbf{1}_{I_2}}{\lambda_0(I_2)}\right) - \beta\left(\frac{\lambda_0 \cdot \mathbf{1}_{I_1}}{\lambda_0(I_1)}\right) \\ \text{s.t. } & s_1 \in I_1, s_2 \in I_2 \\ & \min I_1 \geq \max I_2 \end{aligned}$$

and let $D(s_1, s_2)$ be its value. It is easy to see that

$$0 < D(s_1, s_2) < \beta(s_2) - \beta(s_1).$$

Given this observation, we can in fact replace infimum by minimum in the optimization problem above. Then by Berger's Maximum Theorem, $D(s_1, s_2)$ is continuous. Monotonicity of D is obvious. For each $i \in \{1, 2\}$, let m_i be the induced minimum type by r_i. By Proposition 3.3, we know that $m_1 < m_2$, and thus

$$\begin{aligned} r_1 - r_2 &> \beta(\Lambda(r_1)) - \beta(\Lambda(r_2)) \\ &\geq D(s_1, s_2), \end{aligned}$$

which completes the proof.

Now we proceed to show that $\Gamma(R_n, S)$ has a symmetric PBE. The first step is to obtain ε-PBE of $\Gamma\left(R_n, S\right)$ for arbitrarily small ε.

Lemma B.1. Fix $n \in \mathbb{N}$. For each m, there is $(\rho_m, \Lambda_m) \in \mathcal{T}\left((S, \Sigma_m), R_n, \lambda_0\right) \times \left(\Delta(S)\right)^{R_n}$

such that: (1) $\forall s \in S_m, \forall r \in R_n$,

$$u_s\left(\rho_m(s); \rho_m, \Lambda_m\right) \geq u_s\left(r; \rho_m, \Lambda_m\right)$$

and $\forall r \in R_n$ that is ρ_m-on-path,

$$\Lambda_m(r) = \sum_k \left[\frac{\rho_m(I_{m,k})(r)}{\sum_{k'} \lambda_0(I_{m,k'})\rho_m(I_{m,k'})(r)}\left(\lambda_0 \cdot \mathbf{1}_{I_{m,k}}\right)\right]$$

and thus Λ_m is Bayesian compatible with ρ_m. (2) $\forall \varepsilon > 0$, there is $M_\varepsilon \in \mathbb{N}$ such that $m > M_\varepsilon$ implies $\forall s \in S, \forall r \in R_n$,

$$u_s(\rho_m(s); \rho_m, \Lambda_m) \geq u_s(r; \rho_m, \Lambda_m) - \varepsilon,$$

i.e., (ρ_m, Λ_m) is an ε-PBE of $\Gamma\left(R_n, S\right)$.

Proof We define the following correspondences with domain $\mathcal{T}\left((S, \Sigma_m), R_n, \lambda_0)\right) \times \left(\Delta(S)\right)^{R_n}$:

$$\begin{aligned}
br_s(\rho, \Lambda) &= \arg\max_{r \in R_n} u_s(r; \rho, \Lambda) \\
BR_s(\rho, \Lambda) &= \Delta\left(br_s(\rho, \Lambda)\right) \\
BR_m(\rho, \Lambda) &= \left\{\rho' : \forall k, \rho'(I_{m,k}) \in BR_{s_{m,k}}(\rho, \Lambda)\right\} \\
BU(\rho, \Lambda) &= \left\{\Lambda' : \Lambda' \text{ is Bayesian compatible with } \rho.\ \right\}
\end{aligned}$$

Note that $\Lambda' \in BU(\rho, \Lambda)$ implies that for r that is ρ-on-path,

$$\Lambda'(r) = \sum_k \left[\frac{\rho(I_{m,k})(r)}{\sum_{k'} \lambda_0(I_{m,k'})\rho(I_{m,k'})(r)}\left(\lambda_0 \cdot \mathbf{1}_{I_{m,k}}\right)\right]$$

while for r that is ρ-off-path, $\Lambda'(r)$ can be an arbitrary element in $\Delta(S)$. A fixed point of $BR_m \times BU$ has the desired property in (1), and it suffices to show that $BR_m \times BU$ is u.h.c and compact-and-convex-valued and thus it has a fixed point by Glicksberg Fixed Point Theorem.

It is easy to see that BR_s and BU are u.h.c and compact-and-convex-valued. Note that the mapping φ from $\tilde{\rho} \in \left(\Delta(R)\right)^{S_m}$ to $\rho \in \mathcal{T}\left((S, \Sigma_m), R_n, \lambda_0\right)$ defined by $\varphi\left(\tilde{\rho}\right)\left(I_{m,k}\right) = \tilde{\rho}_{s_{m,k}}$ is continuous, and $BR_m = \varphi \circ \left(\prod_{S_m} BR_s\right)$. Then BR_m has the desired properties.

Since we are working with a finite set R_n, (2) directly follows from the fact that $d_H(S_m, S)$ converges to 0 and $u_s(r;\rho,\Lambda)$ is uniformly continuous in (s,r,ρ,Λ) .

Going to a subsequence if necessary, let $(\rho_n,\Lambda_n) = \lim_{m\to\infty}(\rho_m,\Lambda_m)$. We want to show that ρ_n is a PBE of $\Gamma(R_n,S)$ and thus:

Proposition B.2. For each $n \in \mathbb{N}$, $\Gamma(R_n,S)$ has a symmetric PBE.

Proof (1). Firstly, we show that there is no profitable deviation from ρ_n for almost all $s \in S$, i.e., $\forall B \in \mathcal{B}(S)$ with $\lambda_0(B) > 0$, $\forall r \in R_n$,

$$\int_B u_s(\rho_n;\rho_n,\Lambda_n)\lambda_0(ds) \geq \int_B u_s(r;\rho_n,\Lambda_n)\lambda_0(ds).$$

in which $u_s(\rho_n;\rho_n,\Lambda_n) = \int u_s(r;\rho_n,\Lambda_n)\rho_n(dr|s)$.

By uniform continuity of $u_s(r;\rho,\Lambda)$,

$$\int_B u_s(r;\rho_n,\Lambda_n)\lambda_0(ds) = \lim_{m\to\infty}\int_B u_s(r;\rho_m,\Lambda_m)\lambda_0(ds).$$

Then it suffices to show that

$$\int_B u_s(\rho_n;\rho_n,\Lambda_n)\lambda_0(ds) = \lim_{m\to\infty}\int_B u_s(\rho_m;\rho_m,\Lambda_m)\lambda_0(ds).$$

Note that

$$\begin{aligned}
&|\int_B u_s(\rho_n;\rho_n,\Lambda_n)\lambda_0(ds) - \int_B u_s(\rho_m;\rho_m,\Lambda_m)\lambda_0(ds)| \\
&< |\int_B u_s(\rho_n;\rho_n,\Lambda_n)\lambda_0(ds) - \int_B u_s(\rho_m;\rho_n,\Lambda_n)\lambda_0(ds)| \\
&+ |\int_B u_s(\rho_m;\rho_n,\Lambda_n)\lambda_0(ds) - \int_B u_s(\rho_m;\rho_m,\Lambda_m)\lambda_0(ds)|
\end{aligned}$$

It is easy to see that

$$\int_B u_s(\rho_m;\rho_n,\Lambda_n)\lambda_0(ds) = \lim_{m\to\infty}\int_B u_s(\rho_m;\rho_m,\Lambda_m)\lambda_0(ds).$$

and we claim that

$$\int_B u_s(\rho_n;\rho_n,\Lambda_n)\lambda_0(ds) = \lim_{m\to\infty}\int_B u_s(\rho_m;\rho_n,\Lambda_n)\lambda_0(ds).$$

To see this, since $u_s(r;\rho_n,\Lambda_n)\cdot \mathbf{1}_B(s)$ is a Caratheodory integrand, by definition of $\rho_m \to \rho_n$, we have

$$\begin{aligned}
&\int_B u_s(\rho_m;\rho_n,\Lambda_n)\lambda_0(ds) \\
&= \int_S \left(\int u_s(r;\rho_n,\Lambda_n)\cdot \mathbf{1}_B(s)\cdot \rho_m(dr|s)\right) \\
&\to \int_S \left(\int u_s(r;\rho_n,\Lambda_n)\cdot \mathbf{1}_B(s)\cdot \rho_n(dr|s)\right) \\
&= \int_B u_s(\rho_n;\rho_n,\Lambda_n)\lambda_0(ds)
\end{aligned}$$

(2). Secondly, we show that Λ_n is Bayesian compatible with ρ_n.

Fix $r_0 \in R_n$ and let $e_{r_0}(r) = \mathbf{1}_{r=r_0}$. Note that $e_{r_0}(r)$ is a bounded continuous real-valued function on R_n. Then $\rho_m \to \rho_n$ also implies that if $\int_S \left(\int e_{r_0}(r)\rho_n(dr|s)\right)\lambda_0(ds) > 0$, i.e., r_0 is ρ_n-on-path, then WLOG, $\forall m$, $\int_S \left(\int e_{r_0}(r)\rho_m(dr|s)\right)\lambda_0(ds) > 0$, and $\forall B \in \mathcal{B}(S)$,

$$\begin{aligned}
\Lambda_m(r_0)(B) &= \frac{\int_B \left(\int e_{r_0}(r)\rho_m(dr|s)\right)\lambda_0(ds)}{\int_S \left(\int e_{r_0}(r)\rho_m(dr|s)\right)\lambda_0(ds)} \\
&\to \frac{\int_B \left(\int e_{r_0}(r)\rho_n(dr|s)\right)\lambda_0(ds)}{\int_S \left(\int e_{r_0}(r)\rho_n(dr|s)\right)\lambda_0(ds)}
\end{aligned}$$

Since $\Lambda_m \to \Lambda_n$, we have

$$\Lambda_n(r_0)(B) = \frac{\int_B \left(\int e_{r_0}(r)\rho_n(dr|s)\right)\lambda_0(ds)}{\int_S \left(\int e_{r_0}(r)\rho_n(dr|s)\right)\lambda_0(ds)}$$

which is exactly the Bayesian updating w.r.t ρ_n.

Note that the proofs above relies only on the continuity of u, especially continuity in posteriors. Thus even though the characterization of buyers' behavior for additively separable valuation has loss of generality, existence of symmetric PBE when the space of reserve price is finite may still hold for larger class of buyer valuations (see also Appendix D). However, when the space of reserve price is a continuum, the continuity of u is no longer sufficient, and we require the force of additive separability to establish existence of symmetric PBE for $\Gamma(R,S)$.

Let (ρ_n,Λ_n) be a symmetric PBE of $\Gamma(R_n,S)$; and let ξ_n be the probability measure over $R\times\Delta(S)$ generated by (ρ_n,Λ_n). We would sometimes write $u_s(r,\Lambda_n(r);\xi_n)$ instead of $u_s(r;\rho_n,\Lambda_n)$ to emphasize the importance of ξ_n in payoff convergence. By Proposition 3.3, $\{\rho_n\}$ can be regarded as a sequence of non-decreasing functions from S to R and then by Helly's Selection Theorem, it has an accumulation point in pointwise convergence. Moreover, it is easy to see that ρ_n is essentially a pure strategy profile. And in fact, (ρ^n, Λ^n) is incentive compatible on $\text{supp}(\rho_n)$ for each $s \in S$.

WLOG, let ρ^* be the pointwise limit of $\{\rho_n\}$. Also, let ξ^* be the weak* limit of ξ_n. $r \in R$ is said to be ρ^*-on-path if $\forall \varepsilon > 0$, $(\rho^*)^{-1}\left(\overline{B}_\varepsilon(r)\right)$ is non-empty. Let

$$S(r) = \bigcap_{\varepsilon>0} (\rho^*)^{-1}\left(\overline{B}_\varepsilon(r)\right)$$
$$\hat{\Lambda}(r) = \left\{\lambda \in \Delta(S) : (r, \lambda) \in \text{supp}(\xi^*)\right\}.$$

It is easy to see that $\hat{\Lambda}(r)$ is non-empty iff $S(r)$ is non-empty iff r is ρ^*-on-path. Moreover, by monotonicity of ρ^*, $S(r)$ is connected; and by monotonicity of ρ_n, the support of each element of $\hat{\Lambda}(r)$ is also connected. For r which is ρ^*-on-path, if $\hat{\Lambda}(r)$ contains at least two elements, then ρ^* is unlikely to be an equilibrium. However, with Lemma 3.4, it can be shown that $\hat{\Lambda}(r)$ in fact contains only one element which coincides with Bayesian updating w.r.t ρ^*.

Lemma B.3. For each $r \in R$ which is ρ^*-on-path. $\hat{\Lambda}(r) = \frac{\lambda_0 \cdot \mathbf{1}_{S(r)}}{\lambda_0(S(r))}$.

Proof Fix $\lambda \in \hat{\Lambda}(r)$. Let

$$\underline{s} = \min S(r)$$
$$\overline{s} = \max S(r)$$
$$s_U = \max \text{supp}(\lambda)$$
$$s_L = \max \text{supp}(\lambda).$$

And by definition of $\hat{\Lambda}(r)$, there are $\{r_{n_k}\}$ and $\{s_{L,n_k}\}$ such that: r_{n_k} is ρ_{n_k}-on-path, $r_{n_k} \to r$, $\Lambda_{n_k}(r_{n_k}) \to \lambda$, $s_{L,n_k} \to s_L$ and $\rho_{n_k}(s_{L,n_k}) = r_{n_k}$.

(1). We claim that $\text{supp}\lambda = S(r)$.

To see that $\text{supp}(\lambda) \supset S(r)$, suppose toward a contradiction and WLOG, $s_L > \underline{s}$.

Choose $\varepsilon \in (0, \frac{1}{4}D(\underline{s}, s_L))$. Then there is K such that $k > K$ implies

$$\begin{aligned}\min \operatorname{supp}(\Lambda_{n_k}(r_{n_k})) &> \underline{s} \\ |r_{n_k} - r| &< \varepsilon \\ |\rho_{n_k}(\underline{s}) - r| &< \varepsilon \\ |D(\underline{s}, s_{L,n_k}) - D(\underline{s}, s_L)| &< \varepsilon.\end{aligned}$$

Note that the first inequality follows from the fact that $\operatorname{supp}(\Lambda_{n_k}(r_{n_k}))$ is connected. By the first inequality, $\rho_{n_k}(s_{L,n_k}) > \rho_{n_k}(\underline{s})$; and then by the fourth,

$$|\rho_{n_k}(s_{L,n_k}) - \rho_{n_k}(\underline{s})| > D(\underline{s}, s_{L,n_k}) > D(\underline{s}, s_L) - \varepsilon > 3\varepsilon.$$

However, by the second and the third we have

$$|\rho_{n_k}(s_{L,n_k}) - \rho_{n_k}(\underline{s})| = |r_{n_k} - \rho_{n_k}(\underline{s})| < 2\varepsilon$$

which is a contradiction.

To see that $\operatorname{supp}(\lambda) \subset S(r)$, suppose toward a contradiction and WLOG, $s_L < \underline{s}$. By definition of $\underline{s}$, there is $s_{L,+} \in (s_L, \underline{s})$ such that $\rho^*(s_{L,+}) < r$. Then there is K' such that $k > K'$ implies

$$\rho_{n_k}(s_{L,n_k}) = r_{n_k} > \rho_{n_k}(s_{L,+})$$

and

$$s_{L,n_k} < s_{L,+}$$

contradicting ρ_{n_k} being non-decreasing.

(2). For each pair (s_1, s_2) such that $\underline{s} \leq s_1 < s_2 \leq \bar{s}$ and $\lim \rho_{n_k}(s_1) = \lim \rho_{n_k}(s_2) = r$, we claim that there is K such that $k > K$ implies $\rho_{n_k}([s_1, s_2]) = \{r_{n_k}\}$.

To see this, suppose $\rho_{n_k}(s_1) \neq \rho_{n_k}(s_2)$ for infinitely many k. Then $\rho_{n_k}(s_2) - \rho_{n_k}(s_1) \geq D(s_1, s_2)$ for all these k, contradicting $\lim \rho_{n_k}(s_1) = \lim \rho_{n_k}(s_2) = r$. Then there is K' such that $k > K'$ implies that ρ_{n_k} is constant on $[s_1, s_2]$. Note that for each $k > K'$, if $\rho_{n_k}([s_1, s_2]) \neq r_{n_k}$, we have $|r_{n_k} - \rho_{n_k}([s_1, s_2])| \geq D(s_1, s_2)$. Thus $\rho_{n_k}([s_1, s_2]) \neq r_{n_k}$ can not hold for infinitely many k. Therefore, there is $K > K'$ such that $k > K$ implies $\rho_{n_k}([s_1, s_2]) = \{r_{n_k}\}$.

(3). By (1), if $S(r) = \{s_0\}$, then $\lambda = \delta_{s_0}$. Suppose $S(r)$ is non-degenerate. Fix s_1, s_2 as in (2) and focus on k large enough. Then by (2), if $\rho_{n_k}(\underline{s}) < r_{n_k}$, then $r_{n_k} - \rho_{n_k}(\underline{s}) \geq$

$D(\underline{s}, s_2)$; if $\min \text{supp}(\Lambda_{n_k}(r_{n_k})) < \underline{s}$, then by monotonicity of ρ_{n_k}, $\rho_{n_k}(\underline{s}) = r_{n_k}$. Thus $\{k : \rho_{n_k}(\underline{s}) < r_{n_k}\}$ and $\{k : \rho_{n_k}(\underline{s}) = r_{n_k}\}$ can not be both infinite. A similar statement holds for $\bar{s}$. It follows that $\lambda = \frac{\lambda_0 \cdot \mathbf{1}_{S(r)}}{\lambda_0(S(r))}$.

One implication of this lemma is that if $\rho^*(s) = r$, $\Lambda_n(\rho_n(s)) \to \hat{\Lambda}(r)$. Then we have $u_s\left(\rho_n(s), \Lambda_n(\rho_n(s)); \xi_n\right) \to u_s\left(r, \hat{\Lambda}(r); \xi^*\right)$. Given $\hat{\Lambda}$, it remains to pin down belief updating for off-path reserve prices. Firstly, we extend Λ_n which has domain R_n to R: for each $r \in R$, let

$$\kappa(r) = \arg\min_{r' \in R_n} d(r, r')$$
$$\overline{\Lambda}_n(r) = \Lambda_n\left(\kappa(r)\right).$$

Note that $\overline{\Lambda}_n$ is a correspondence with compact graph and it is single-valued except for finitely many off-path reserve prices. Moreover, by uniform continuity of $u(\cdot)$, $\forall \varepsilon > 0$, there is N_ε such that $n > N_\varepsilon$ implies $\forall s \in S$,

$$\max_r \left\{\max u_s\left(r, \overline{\Lambda}_n(r)\ \xi_n\right)\right\} < u_s\left(\rho_n(s), \overline{\Lambda}_n(\rho_n(s)); \xi_n\right) + \varepsilon$$

Then $\left\{(\rho_n, \overline{\Lambda}_n)\right\}$ is asymptotically ε-sequentially-rational.

Secondly, we identify $\overline{\Lambda}_n$ with its graph $Gr(\overline{\Lambda}_n)$ and regard $Gr(\overline{\Lambda}_n)$ as an element in $K_{R\times\Delta(S)}$, the space of compact subsets of $R \times \Delta(S)$. Since $K_{R\times\Delta(S)}$ topologized by a Hausdorff distance is compact, WLOG let $Gr(\overline{\Lambda}^*)$ be the limit of $\left\{Gr(\overline{\Lambda}_n)\right\}$ and let $\overline{\Lambda}^*$ be the correspondence induced by $Gr(\overline{\Lambda}^*)$. Note that for r that is ρ^*-on-path, $\hat{\Lambda}(r) \in \overline{\Lambda}^*$. The next lemma states that any selection from $\overline{\Lambda}^*$ that is Bayesian compatible with ρ^* constitutes a symmetric PBE.

Lemma B.4. $\forall s \in S$,

$$\max_{(r,\lambda)\in Gr(\overline{\Lambda}^*)} u_s(r, \lambda; \xi^*) \leq u_s\left(\rho^*(s), \hat{\Lambda}(\rho^*(s)); \xi^*\right).$$

Proof Suppose not. Let $s_0 \in S, (r_0, \lambda_0) \in Gr(\overline{\Lambda}^*)$ be such that

$$u_{s_0}(r_0, \lambda_0; \xi^*) > u_{s_0}\left(\rho^*(s_0), \overline{\Lambda}(\rho^*(s_0); \xi^*\right)$$

Let

$$0 < \varepsilon < \frac{u_{s_0}(r_0, \lambda_0; \xi^*) - u_{s_0}\left(\rho^*(s_0), \hat{\Lambda}(\rho^*(s_0)); \xi^*\right)}{4}.$$

Then there is N_ε such that $n > N_\varepsilon$ implies: there is $(r', \lambda') \in Gr(\overline{\Lambda}_n)$ such that

$$u_{s_0}(r', \lambda'; \xi_n) > u_{s_0}(r_0, \lambda_0; \xi^*) - \varepsilon$$
$$u_{s_0}\left(\rho_n(s), \Lambda_n(\rho_n(s)); \xi_n\right) < u_{s_0}\left(\rho^*(s_0), \hat{\Lambda}(\rho^*(s_0)); \xi^*\right) + \varepsilon$$

which then imply

$$u_{s_0}(r', \lambda'; \xi_n) > u_{s_0}\left(\rho_n(s), \Lambda_n(\rho_n(s)); \xi_n\right) + 2\varepsilon$$

contradicting asymptotic ε-sequential-rationality of $\left\{(\rho_n, \overline{\Lambda}_n)\right\}$.

Now let $\Lambda^* : R \to \Delta(S)$ be such that: (1). $\forall r \in R$, $\Lambda^*(r) \in \overline{\Lambda}^*(r)$; (2). $\forall r$ that is ρ^*-on-path, $\Lambda^*(R) = \hat{\Lambda}(r)$. Lemma B.3 and Lemma B.4 imply that (ρ^*, Λ^*) constitutes a symmetric PBE, and we have:

Theorem 1. $\Gamma(R, S)$ has a symmetric PBE.

C Non-existence of separating limit equilibrium

Recall that $t_* = \alpha^{-1}(c(\underline{s}) - \beta(\underline{s}))$, and consider the initial value problem:

$$\begin{aligned}
\frac{d\sigma}{dq} &= \frac{1}{1-q} \cdot \frac{\alpha(\mu) + \beta(\sigma) - c(\sigma)}{\beta'(\sigma)} \\
\frac{d\mu}{dq} &= \frac{1}{q} \cdot \frac{\alpha(t) - \alpha(\mu)}{\alpha'(\mu)} \\
\frac{dt}{dq} &= \frac{1}{q} \cdot \frac{G(\sigma)}{kf(t)}. \\
(q_0, \sigma_0, \mu_0, t_0) &= (0, \underline{s}, t_*, t_*).
\end{aligned} \tag{$*$}$$

The proof of $(\sigma, \mu, t)(q) = (\underline{s}, t_*, t_*)$ being the unique solution to ($*$) on $[0, 1)$ relies on a minor modification of the technique in Gard (1978). For each $\mathcal{C}^\infty$-function $y : (0, 1) \to$

$\mathbb{R}^N$, we write $y^{(n)} = \frac{d^n y}{dq^n}$. Also let $x = (\sigma, \mu, t)$ and express ($*$) in the form

$$x'(q) = H(q, x(q)), q \in (0,1)$$
$$x(0) = x_0. \tag{$*'$}$$

Note that if a solution to the problem ($*'$) passes through (q, x_0) with $q > 0$, then it coincides with the constant solution $x(q) = x_0$. Therefore, to establish uniqueness, it suffices to study solutions for a small neighborhood of $(0, x_0)$. Let

$$B_\varepsilon^+(0, x_0) = \left\{(q, x) \geq (0, x_0) : \|(q, x) - (0, x_0)\| < \varepsilon\right\}$$

be the set of interest.

Lemma C.1. Fix $\varepsilon \in (0,1)$. Suppose (1) there is $M \in \mathbb{N}$ such that $\forall (q, x), (q, y) \in B_\varepsilon^+(0, x_0)$ with $q > 0$,

$$(H(q, x) - H(q, y)) \cdot (x - y) \leq \frac{M}{q}\|x - y\|^2,$$

and (2) for each solution $x(q)$ to the initial value problem ($*'$) and each $n \leq M$,

$$\lim_{q \to 0^+} x^{(n)}(q) = \mathbf{0}.$$

Then ($*'$) has at most one solution on $[0, \varepsilon)$.

Proof Let $x(q)$ and $y(q)$ be solutions to ($*'$). Define

$$v(q) = \begin{cases} \frac{1}{2}\left[\|x(q) - y(q)\| \cdot q^{-M}\right]^2, & \text{if } q > 0 \\ 0, & \text{if } q = 0. \end{cases}$$

Note that for each $i \in N$, by applying L'Hospital's rule M times,

$$\lim_{q \to 0^+} \frac{x_i(q) - y_i(q)}{q^M} = \frac{x^{(M)}(0^+) - y^{(M)}(0^+)}{M!} = 0.$$

Then $v(q)$ is continuous on $[0,1)$. Now for each $q > 0$, we have

$$v'(q) = -\frac{\|x(q) - y(q)\|^2}{q^{2M}}\frac{M}{q} + \sum_{i=1}^{N} \frac{\left(H_i(q,x(q)) - H_i(q,y(q))\right)(x_i(q) - y_i(q))}{q^{2M}} \leq 0,$$

by hypothesis (1). Since $v(q)$ is non-negative, $v \equiv 0$ on $[0,1)$ and thus $x(q) = y(q)$.

Hypothesis (1) in Lemma C.1 follows from the fact that by our assumptions, $qH(q,x)$ is Lipschitz in x: there is $M \in \mathbb{N}$ such that $\forall x, y$, $\|qH(q,x) - qH(q,y)\| \leq M\|x - y\|$. Then

$$\begin{aligned}(qH(q,x) - qH(q,y)) \cdot (x - y) &\leq \|qH(q,x) - qH(q,y)\| \cdot \|x - y\| \\ &\leq M\|x - y\|^2.\end{aligned}$$

and thus $(H(q,x) - H(q,y)) \cdot (x - y) \leq \frac{M}{q}\|x - y\|^2$.

Now we proceed in three steps. First imposing that for each $n \in \mathbb{N}$, $\lim_{q\to 0^+} x^{(n)}(q)$ exists (finite or infinite), we show that this regularity condition implies Hypothesis (2) of Lemma C.1. Second we show that $\lim_{q\to 0^+} x^{(n)}(q)$ exists for $n \in \{1,2\}$ and the local Lipschitz bound of $qH(q,x)$ at $q = 0$ is determined by $\frac{g(\underline{s})}{kf(t_*)}$. Theorem 2 follows immediately from these two steps.

C.1 Uniqueness with regularity

With regularity imposed, let

$$x^n(0^+) = \lim_{q\to 0^+} x^{(n)}(q).$$

We want to show that for each $n \in \mathbb{N}$, $x^n(0^+) = 0$

Suppose (σ, μ, t) is a solution to ($*$) and we proceed by induction. Recall that

$$\sigma^{(1)} = \frac{1}{1-q} \cdot \frac{\alpha(\mu) + \beta(\sigma) - c(\sigma)}{\beta^{(1)}(\sigma)} \tag{1_σ}$$

$$\mu^{(1)} = \frac{1}{q} \cdot \frac{\alpha(t) - \alpha(\mu)}{\alpha^{(1)}(\mu)} \tag{1_μ}$$

$$t^{(1)} = \frac{1}{q} \cdot \frac{G(\sigma)}{kf(t)}. \tag{1_t}$$

$$(q_0, \sigma_0, \mu_0, t_0) = (0, \underline{s}, t_*, t_*).$$

It follows from (1_σ) that $\sigma^{(1)}(0^+) = 0$. Applying L'Hospital's rule to (1_t) yields

$$t^{(1)}(0^+) = \frac{g(\underline{s})}{kf(t_*)} \cdot \sigma^{(1)}(0^+) = 0.$$

Then applying L'Hospital's rule to (1_μ) yields

$$\mu^{(1)}(q) = O(q) - \mu^{(1)}(q)$$

and thus $\mu^{(1)}(0^+) = 0$.

Now suppose

$$\forall n \leq \hat{n},\ (\sigma^{(n)}, \mu^{(n)}, t^{(n)})(0^+) = \mathbf{0}. \tag{$\hat{n}$}$$

The remaining inductive steps may be heavy in notations, but the idea is simple and the argument follows the same logic as in the case $n = 1$: the behavior of $\sigma^{(\hat{n}+1)}$ when q approaches zero is determined by the behavior of lower order derivatives; with this additional knowledge about $\sigma^{(\hat{n}+1)}$, the behavior of $t^{(\hat{n}+1)}$ is determined; finally, with the knowledge about $t^{(\hat{n}+1)}$, the behavior of $\mu^{(\hat{n}+1)}$ is determined.

We start the inductive steps from $\sigma(q)$. To circumvent tedious calculation, we introduce the following formality: let $P_{\sigma,\mu}[d_\sigma, d_\mu]$ be the collection of functions of the form

$$\sum_{i<\infty} a_i(\sigma, \mu) \left(\prod_{1 \leq j \leq d_\sigma} (\sigma^{(j)})^{m_{i,j}} \prod_{1 \leq j \leq d_\mu} (\mu^{(j)})^{n_{i,j}} \right)$$

such that: $m_{i,j}, n_{i,j} \in \mathbb{N}$; and for each i, a_i is $\mathcal{C}^\infty$ and non-zero, and there is j with $m_{i,j} > 0$ or $n_{i,j} > 0$. In other words, $P_{\sigma,\mu}[d_\sigma, d_\mu]$ contains the polynomials with zero constant over $\left\{\sigma^{(j)}, \mu^{(j')}\right\}_{j \leq d_\sigma, j' \leq d_\mu}$. Note that d_σ and d_μ are the maximum orders of derivatives. Given (1_σ), we have

$$\sigma^{(2)} \in \frac{P_{\sigma,\mu}[1,1]}{1-q} + \frac{\sigma^{(1)}}{1-q} = \frac{P_{\sigma,\mu}[1,1]}{1-q}.$$

Then by induction, for each $n \geq 2$, $\sigma^{(n)} \in \frac{P_{\sigma,\mu}[n-1,n-1]}{1-q}$. In particular, $\sigma^{(\hat{n}+1)} \in \frac{P_{\sigma,\mu}[\hat{n},\hat{n}]}{1-q}$. Then by the induction hypothesis ($\hat{n}$) and the definition of $P_{\sigma,\mu}[\hat{n},\hat{n}]$, $\sigma^{(\hat{n}+1)}(0^+) = 0$.

Now we proceed to analyze $t(q)$. Denote $G(\sigma)$ as $\sigma^{(0)}$, and let $Q_{\sigma,t}[d_\sigma, d_t; \overline{d}_q]$ be the collection of functions of the form

$$\sum_{i<\infty} \frac{a_i(\sigma, t)}{q^{l_i}} \left(\prod_{0 \leq j \leq d_\sigma} (\sigma^{(j)})^{m_{i,j}} \prod_{1 \leq j \leq d_t} (t^{(j)})^{n_{i,j}} \right)$$

such that: $l_i, m_{i,j}, n_{i,j} \in \mathbb{N}$; for each i, a_i is $\mathcal{C}^\infty$ and non-zero, and there is j with $m_{i,j} > 0$. Moreover, for each i, if j is the largest integer such that $m_{i,j} > 0$, then $l_i + j \leq \overline{d}_q$. Note that $\overline{d}_q$ is a control to guarantee that the convergence of q^{l_i} to zero can be "absorbed" by sufficiently low order derivatives of σ. We already know from (1_t) that $t^{(1)} \in Q_{\sigma,t}[0,0;1]$. Then by induction, for each $n \geq 1$, $t^{(n)} \in Q_{\sigma,t}[n-1, n-1; n]$. In particular, $t^{(\hat{n}+1)} \in Q_{\sigma,t}[\hat{n}, \hat{n}; \hat{n}+1]$. Since we have established that $\sigma^{(\hat{n}+1)}(0^+) = 0$, by L'Hospital's rule, for each $n \leq \hat{n}+1$

$$\lim_{q \to 0^+} \frac{\sigma^{(n-l)}(q)}{q^l} = \sigma^{(n)}(0^+) = 0.$$

Therefore, $t^{(\hat{n}+1)}(0^+) = 0$.

Finally, we come to $\mu(q)$. Consider the following equivalent form of (1_μ):

$$\alpha^{(1)}(\mu)\, \mu^{(1)} = \frac{\alpha(t) - \alpha(\mu)}{q},$$

applying $\frac{d}{dq}$ to which yields,

$$\begin{aligned} \alpha^{(2)}(\mu)\, (\mu^{(1)})^2 + \alpha^{(1)}(\mu)\, \mu^{(2)} &= \frac{\alpha^{(1)}(t)\, t^{(1)} - \alpha^{(1)}(\mu)\, \mu^{(1)}}{q} - \frac{\alpha(t) - \alpha(\mu)}{q^2} \\ &= \frac{\alpha^{(1)}(t)\, t^{(1)} - 2\alpha^{(1)}(\mu)\, \mu^{(1)}}{q} \end{aligned} \tag{2_μ}$$

Let $P_\mu[d_\mu]$ be the collection of functions of the form

$$\sum_{i<\infty} a_i(\mu) \left(\prod_{1 \leq j \leq d_\mu} (\mu^{(j)})^{m_{i,j}} \right)$$

such that: $m_{i,j} \in \mathbb{N}$; and for each i, a_i is $\mathcal{C}^\infty$ and non-zero, and there is j with $m_{i,j} > 0$. Let $P_t[d_t]$ be defined similarly. Then we write (2_μ) as

$$P_\mu[1] + \alpha^{(1)}(\mu)\, \mu^{(2)} \sim \frac{P_t[1] - 2\alpha^{(1)}(\mu)\, \mu^{(1)}}{q}.$$

It is easy to see that given $\mu^1(0^+) = t^2(0^+) = 0$, $\mu^2(0^+) = 0$. Applying $\frac{d}{dq}$ to (2_μ), we also have

$$P_\mu[2] + \alpha^{(1)}(\mu)\,\mu^{(3)} \sim \frac{P_t[2] + P_\mu[1] - 2\alpha^{(1)}(\mu)\,\mu^{(2)}}{q} - \frac{P_\mu[1] + \alpha^{(1)}(\mu)\,\mu^{(2)}}{q}$$
$$\sim \frac{P_t[2] + P_\mu[1] - 3\alpha^{(1)}(\mu)\,\mu^{(2)}}{q}.$$

Then by induction, for each $n \geq 3$,

$$P_\mu[n-1] + \alpha^{(1)}(\mu)\,\mu^{(n)} \sim \frac{P_t[n-1] + P_\mu[n-2] - n\alpha^{(1)}(\mu)\,\mu^{(n-1)}}{q}.$$

In particular, for $\hat{n} \geq 3$,

$$P_\mu[\hat{n}] + \alpha^{(1)}(\mu)\,\mu^{(\hat{n}+1)} \sim \frac{P_t[\hat{n}] + P_\mu[\hat{n}-1] - (\hat{n}+1)\alpha^{(1)}(\mu)\,\mu^{(\hat{n})}}{q}. \qquad ((\hat{n}+1)'_\mu)$$

Then given the induction hypothesis and $t^{(\hat{n}+1)}(0^+) = 0$,

$$\lim_{q\to 0^+} \frac{P_\mu[\hat{n}-1]}{q} = \lim_{q\to 0^+} P_\mu[\hat{n}] = 0$$
$$\lim_{q\to 0^+} \frac{P_t[\hat{n}]}{q} = \lim_{q\to 0^+} P_t[\hat{n}+1] = 0.$$

Then by L'Hospital's rule,

$$\mu^{(\hat{n}+1)}(q) = o(q) - (\hat{n}+1)\mu^{(\hat{n}+1)}(q)$$

and thus $\mu^{(\hat{n}+1)}(0^+) = 0$, which completes the proof.

C.2 Uniqueness with large buyer-seller ratio

From the argument in the previous subsection, we know that $\sigma^{(1)}(0^+) = t^{(1)}(0^+) = 0$. Also, if $\mu^{(1)}(0^+)$ exists, then it is zero, and $\sigma^{(2)}(0^+) = t^{(2)}(0^+) = 0$. We want to show that $\mu^{(1)}(0^+)$ and $\mu^{(2)}(0^+)$ exist.

Note that

$$\begin{aligned}\limsup_{q\to 0^+}\mu^{(1)} &= \limsup_{q\to 0^+}\left(\frac{1}{q}\cdot\frac{\alpha(t)-\alpha(\mu)}{\alpha^{(1)}(\mu)}\right)\\ &\le \limsup_{q\to 0^+}\left(\frac{\alpha^{(1)}(t)\,t^{(1)}-\alpha^{(1)}(\mu)\,\mu^{(1)}}{\alpha^{(1)}(\mu)}\right)\\ &= -\liminf_{q\to 0^+}\mu^{(1)},\end{aligned}$$

then we have

$$\limsup_{q\to 0^+}\mu^{(1)}+\liminf_{q\to 0^+}\mu^{(1)}\le 0.$$

Since in a separating limit equilibrium $\mu(q)$ is strictly increasing, we also have

$$\liminf_{q\to 0^+}\mu^{(1)}\ge 0.$$

As a result, $\mu^{(1)}(0^+)=0$.

The argument for $\mu^{(2)}(0^+)$ is similar. Recall that

$$\alpha^{(2)}(\mu)\,(\mu^{(1)})^2+\alpha^{(1)}(\mu)\,\mu^{(2)}=\frac{\alpha^{(1)}(t)\,t^{(1)}-2\alpha^{(1)}(\mu)\,\mu^{(1)}}{q}.$$

Then taking $\limsup_{q\to 0^+}$ on both sides and eliminating $\alpha^1(t_*)$ yields

$$\limsup_{q\to 0^+}\mu^{(2)}=-2\liminf_{q\to 0^+}\left(\frac{\mu^{(1)}}{q}\right)\le 0.$$

On the other hand, we also have

$$\begin{aligned}\liminf_{q\to 0^+}\mu^{(2)} &= -2\limsup_{q\to 0^+}\left(\frac{\mu^{(1)}}{q}\right)\\ &\ge -2\limsup_{q\to 0^+}\mu^{(2)},\end{aligned}$$

and thus

$$\liminf_{q\to 0^+}\mu^{(2)}+2\limsup_{q\to 0^+}\mu^{(2)}\ge 0.$$

As a result, $\mu^{(2)}(0^+)=0$.

Finally, let $z(q,x) = qH(q,x)$. Fix a pair (q,x_1) and (q,x_2) in $B_\varepsilon^+(0,x_0)$. We have

$$
\begin{aligned}
|z_{\sigma,1} - z_{\sigma,2}| &\leq \frac{q}{1-q} \cdot M_1 \|x_1 - x_2\| \\
|z_{\mu,1} - z_{\mu,2}| &\leq |\alpha(t_1) - \alpha(\mu_1))| \cdot |\frac{1}{\alpha^{(1)}(\mu_1)} - \frac{1}{\alpha^{(1)}(\mu_2)}| \\
&\quad + \frac{1}{\alpha^{(1)}(\mu_2)} \left(|\alpha(t_1) - \alpha(t_2)| + |\alpha(\mu_1) - \alpha(\mu_2)| \right) \\
|z_{t,1} - z_{t,2}| &\leq \frac{1}{k} \left(G(\sigma_1) |\frac{1}{f(t_1)} - \frac{1}{f(t_2)}| + \frac{|G(\sigma_1) - G(\sigma_2)|}{f(t_2)} \right).
\end{aligned}
$$

Therefore, for each $\eta > 0$, there is ε sufficiently small such that

$$
\begin{aligned}
\|z_1 - z_2\|^2 \leq &\left[\left(\frac{g(\underline{s})}{kf(t_*)} \right)^2 + \eta^2 \right] (\sigma_1 - \sigma_2)^2 \\
&+ (2 + \eta^2) \left[(\mu_1 - \mu_2)^2 + (t_1 - t_2)^2 \right]
\end{aligned}
$$

and thus

$$
\|z_1 - z_2\| \leq \left(\max \left\{ \sqrt{2}, \frac{g(\underline{s})}{kf(t_*)} \right\} + \eta \right) \|x_1 - x_2\|.
$$

In particular, when $k > \frac{g(\underline{s})}{2f(t_*)}$, we have $\|z_1 - z_2\| \leq 2\|x_1 - x_2\|$ on $B_\varepsilon^+(0,x_0)$ for ε sufficiently small.

D Alternative buyer valuations

Here we characterize buyer's participation/search behavior in a finite market model with two sellers and heterogeneous objects. In particular, we allow a buyer's valuation of one object and that of the other to differ in a more general manner, containing the additively separable valuation we have analyzed as a special case. The main message from this exercise is that quality can still affect buyer's participation/search behavior after controlled for induced minimum type. Interestingly, there may be quality searching types in the sense that their equilibrium behavior is mainly driven by quality and is independent of small changes in minimum types. However, this interesting feature is detrimental to tractability. Even under the limited generality considered below, we lose the strong monotonicity result of Proposition 3.3, and we cannot establish existence of symmetric PBE when the space of reserve price is a continuum

Let $J = A, B$ be the set of sellers/objects, and suppose there are N buyers. For each $i \in A, B$, a buyer of type θ evaluates object i according to a continuously differentiable function $v_i(\theta)$, and $\forall \theta \in \Theta$, $v_i'(\theta) > 0$. Then given r_i, we can define the induced minimum type m_i as the unique solution to $v_i(m_i) = r_i$. Let $\phi(\theta) = \left(\frac{v_B'(\theta)}{v_A'(\theta)}\right)^{\frac{1}{N-1}}$. For tractability, we impose the following assumptions on $\{v_i\}$:

Assumption D. $\frac{1+F(\theta)}{1+\phi(\theta)}$ and $\frac{1+F(\theta)}{1+\frac{1}{\phi(\theta)}}$ are increasing in θ;

Note that additively separable valuation corresponds to the special case where $v_A(\theta) - v_B(\theta) \equiv$ constant and $\phi(\theta) = 1$. On the other hand, if v_i is derived from a multiplicatively separable valuation $v(\theta, s_i) = a(\theta) \cdot b(\lambda_i)$, then $\phi(\theta) \equiv$ constant and Assumption D is satisfied.

WLOG, consider $m_A \leq m_B \leq 1$. Let $p_i(\theta; m_A, m_B)$ be the probability that a type-θ buyer participates in auction i. For $\theta \geq m_i$, it can be shown by integration by parts that her interim payoff from auction i is:

$$V_i(\theta; m_A, m_B) \int_{m_i}^{\theta} \left[1 - \int_x^1 p_i(y; m_A, m_B) dF(y)\right]^{N-1} dv_i(x).$$

Let $\Delta V = V_A - V_B$ and in equilibrium ΔV is continuously differentiable. We subtract $\boldsymbol{m}$ from notations when there is no confusion. Suppose there is an open interval I such that $\Delta V \equiv 0$ on I, then for each $\theta \in I$, the probability that a type-θ buyer participates in auction A is given by:

$$\hat{p}(\theta) = \left(\frac{\phi'}{(1+\phi)^2} \frac{1+F}{f} + \frac{\phi}{1+\phi}\right)(\theta)$$

and Assumption D guarantees that $\hat{p}(\theta) \in (0, 1)$.

Under Assumption D, we show that buyer's participation/search behavior is uniquely determined and has the following segmented form: there is $\tau^* \le \theta^* \in [0,1]$ such that:

$$p_A(\theta;\tau^*,\theta^*) = \begin{cases} 1, & \text{if } \theta \in [m_A, \tau^*) \\ \hat{p}(\theta), & \text{if } \theta \in [\tau^*, \theta^*] \\ \mathbf{1}_{\phi(1)<1}, & \text{if } \theta > \theta^*. \end{cases} \tag{D.0}$$

and $p_B(\theta) = 1 - p_A(\theta)$ on $[m_A, 1]$. Note that in the case where $v_A(\theta) - v_B(\theta)$ is constant, $\hat{p}(\theta) = \frac{1}{2}$ and $\theta^* = 1$. In general, if $\theta^* \in (\tau^*, 1)$, then $\theta^* < 1$ is locally independent of $\boldsymbol{m}$. For example, when v_i is derived from a multiplicatively separable valuation, $\phi(1) < 1$ if and only if object A has a higher quality. In this sense, a buyer of type in $(\theta^*, 1]$ behaves as if she is a quality searcher.

The proof starts with an analysis of ΔV in equilibrium. The first lemma states that the indifference interval $\{\theta : \Delta V(\theta) = 0\}$ is connected.

Lemma D.1. If $\Delta V(\theta) < (>)0$ and $\Delta V' = 0$, then θ is a strict local maximum(minimum) of ΔV. Contrapositively, if $\Delta V(\theta) < (>)0$, then θ is not a local maximum of ΔV. Therefore, $\{\theta : \Delta V(\theta) = 0\}$ is connected.

Proof Here we only present the proof for $\Delta V(\theta) < 0$. The argument for the other case is similar. Note that

$$\operatorname{sgn}\left(\Delta V'(\theta)\right) = -\operatorname{sgn}\left(\int_\theta^1 p_A(x)dF(x) - \frac{1-\phi(\theta)F(\theta)}{1+\phi(\theta)}\right) \tag{D.1}$$

By assumption,

$$\int_\theta^1 p_A(x)dF(x) - \frac{1-\phi(\theta)F(\theta)}{1+\phi(\theta)} = 0.$$

Since $\Delta V(\theta) < 0$, $p_A \equiv 0$ in a neighbourhood of θ. And we have,

$$\text{LHS}' = \frac{\phi(1+F)}{(1+\phi)^2} \cdot \left(\frac{\phi'}{\phi} + \frac{f}{1+F}(1+\phi)\right) > 0$$

Therefore, θ is a strict local maximum of ΔV.

Now we proceed to characterize the upper end of the indifference interval. Note that another implication of the equation D.1 is that if $\phi(\theta) < F(\theta)$, then $\Delta V'(\theta) > 0$; similarly, if $\phi(\theta) > 1/F(\theta)$, then $\Delta V'(\theta) < 0$. And such θ is not in the indifference interval. The second lemma states that $\text{Gr}(\phi)$ and $\text{Gr}(F \cup 1/F) = \text{Gr}(F) \cup \text{Gr}(1/F)$ (restricted to $\theta \in [m_B, 1]$) intersect in a simple pattern, which renders tractability.

Lemma D.2. $\text{Gr}(\phi) \cap \big(\text{Gr}(F \cup 1/F)\big)$ is a singleton.

Proof Since ϕ is continuous, $\text{Gr}(\phi) \cap \big(\text{Gr}(F \cup 1/F)\big)$ is non-empty. Here we show that $\text{Gr}(\phi)$ and $Gr(F)$ have at most one intersection. By assumption, for each θ,

$$\left(\frac{1+F}{1+\phi}\right)' = \frac{f(1+\phi) - \phi'(1+F)}{(1+\phi)^2} > 0$$

and thus if y is a zero of $\phi - F$, then $\phi'(y) < f(y)$. Suppose $\theta_1 < \theta_2$ are two zeros of $\phi - F$, then there is $\varepsilon_1, \varepsilon_2 > 0$ such that $\theta_1 + \varepsilon_1 < \theta_2 - \varepsilon_2$, $(\phi - F)(\theta_1 + \varepsilon_1) < 0$, and $(\phi - F)(\theta_2 - \varepsilon_2) > 0$. Then there is $\theta' \in (\theta_1 + \varepsilon_1, \theta_2 - \varepsilon_2)$ such that $(\phi - F)(\theta') < 0$ and $\big(\phi - F\big)'(\theta') > 0$, but then $\left(\frac{1+F}{1+\phi}\right)'$ at θ' is negative, which is a contradiction. A similar argument will establish that $\text{Gr}(\phi)$ and $\text{Gr}(1/F)$ have at most one intersection, which completes the proof.

Lemma D.2 immediately allows us to characterize when seller A reaps the whole market.

Corollary D.3. Let

$$\Delta\tilde{V}(\theta) = \int_{m_A}^{\theta} (F(y))^{N-1} dv_A(y) - (v_B(\theta) - v_B(m_B)).$$

Restricted to $[m_B, 1]$:

1. if $\text{Gr}(\phi) \cap \text{Gr}(F) = \emptyset$, then $p_A \equiv 1$ if and only $\Delta\tilde{V}(1) \geq 0$;
2. if $\text{Gr}(\phi) \cap \text{Gr}(F) = \big\{(\theta^*, F(\theta^*))\big\}$, then $p_A \equiv 1$ if and only $\Delta\tilde{V}(\theta^*) \geq 0$ or $m_A > \theta^*$.

Proof Note that $p_A \equiv 1$ if and only if $\Delta\tilde{V}(\theta) \geq 0$ on $[m_B, 1]$. The statement follows from Lemma D.2 and the fact that $\text{sgn}(\Delta\tilde{V}) = -\text{sgn}(\phi - F)$.

Throughout the remaining part of this section, we focus on the cases when A can not reap the whole market in equilibrium. The next lemma states that the upper end of the indifference interval is almost determined by the intersection of $\mathrm{Gr}(\phi)$ and $\mathrm{Gr}(F) \cup \mathrm{Gr}(1/F)$.

Lemma D.4. Suppose $p_A \equiv 1$ on $[m_A, 1]$ is not an equilibrium, we have:

1. if $\mathrm{Gr}(\phi) \cap \mathrm{Gr}(F \cup 1/F) = \left\{(\theta^1, F(\theta^1))\right\}$, then $\Delta V(\theta^1) = 0$ and $\Delta V'(\theta^1) = 0$.
2. if $\mathrm{Gr}(\phi) \cap \mathrm{Gr}(F \cup 1/F) = \left\{(\theta^2, (1/F)(\theta^2)\right\}$, then $\Delta V(\theta^2) \geq 0$. If $\Delta V(\theta^2) = 0$, then $\Delta V'(\theta^2) = 0$.

Proof (1). By Lemma D.2, for each $\theta > \theta^1$, we have $\phi(\theta) < F(\theta)$ and $\Delta V'(\theta) < 0$. By Corollary D.3, $\theta^1 > m_B$. Suppose $\Delta V(\theta^1) < 0 \leq \Delta V(m_B)$. Then there is $\theta^a \in [m_B, \theta^1]$ such that $\Delta V(\theta^a) < 0$ and θ^a is a local minimum of ΔV, contradicting Lemma D.1. Now suppose $\Delta V(\theta^1) > 0$. Again by Lemma D.2, for each $\theta < \theta^1$ $\phi(\theta) > F(\theta)$ and $\Delta V'(\theta) < 0$. Then there is $\theta^b \in [m_B, \theta^1]$ such that $\phi(\theta^b) > F(\theta^b)$, $\Delta V(\theta^b) \geq 0$, and $\Delta V > 0$ on $(\theta^b, 1]$. However, $\Delta V > 0$ on $(\theta^b, 1]$ implies that

$$\begin{aligned}
\mathrm{sgn}(\Delta V'(\theta^b)) &= -\mathrm{sgn}\left(\int_{\theta^b}^1 p_A(x)dF(x) - \frac{1-\phi(\theta^b)F(\theta^b)}{1+\phi(\theta^b)}\right) \\
&= -\mathrm{sgn}\left(1 - F(\theta^b) - \frac{1-\phi(\theta^b)F(\theta^b)}{1+\phi(\theta^b)}\right) \\
&< 0.
\end{aligned}$$

which is a contradiction. Therefore, $\Delta V(\theta^1) = 0$, and $\Delta V'(\theta^1) = 0$ follows from Lemma D.1. (2). By Lemma D.2, for each $\theta > \theta^2$, $\phi(\theta) > 1/F(\theta)$ and $\Delta V'(\theta) < 0$. Suppose $\Delta V(\theta^2) < 0$, then there is $\theta^c \in [m_B, \theta 2]$ such that $\phi(\theta^c) < 1/F(\theta^c)$, $\Delta V'(\theta^c) \leq 0$, and $\Delta V < 0$ on $(\theta^c, 1]$. Similar to the argument in (1), $\Delta V < 0$ on $(\theta^c, 1]$ implies that

$$\begin{aligned}
\mathrm{sgn}(\Delta V'(\theta^c)) &= -\mathrm{sgn}\left(\int_{\theta^c}^1 p_A(x)dF(x) - \frac{1-\phi(\theta^c)F(\theta^c)}{1+\phi(\theta^c)}\right) \\
&= -\mathrm{sgn}\left(0 - \frac{1-\phi(\theta^c)F(\theta^c)}{1+\phi(\theta^c)}\right) \\
&> 0,
\end{aligned}$$

which is a contradiction. When $\Delta V(\theta^2) = 0$, note that $p_A = 0$ on $[\theta^2, 1]$, then

$$\int_{\theta^2}^1 p_A(x)dF(x) = 0 = \frac{1-\phi(\theta^2)F(\theta^2)}{1+\phi(\theta^2)}$$

implying that $\Delta V(\theta^2) = 0$.

In case (1), the upper end of the indifference interval is θ^1. In case (2), either the upper end is θ^2, or the indifference interval is degenerate. Now it remains for us to determine the lower end of the indifference interval.

Define $p_A(\theta; t, y)$ as in (D.0), and let

$$h(t;y) = \int_{m_A}^{t} \left(1 - \int_x^1 p_A(z;t,y)dF(z)\right)^{N-1} dv_A(x) - \int_{m_B}^{t} \left(F(x) + \int_x^1 p_A(z;t,y)dF(z)\right)^{N-1} dv_B(x)$$

be the payoff difference between auction A and auction B for a type-t buyer if all other buyers follow the strategy described by $p_A(\theta; t, y)$. The following proposition completes the characterization of buyer's participation/search behavior.

Proposition D.5. Suppose $p_A \equiv 1$ on $[m_A, 1]$ is not an equilibrium, then there is a unique equilibrium in which:

1. if $\text{Gr}(\phi) \cap \left(\text{Gr}(F \cup 1/F)\right) = \left\{(\phi(\theta^1), F(\theta^1))\right\}$, then there is a unique $\tau^1 \in [m_B, \theta^1]$ such that $h(\tau^1, \theta^1) = 0$, and $p_A^*(\theta) = p_A(\theta; \tau^1, \theta^1)$;
2. if $\text{Gr}(\phi) \cap \left(\text{Gr}(F \cup 1/F)\right) = \left\{(\phi(\theta^2), 1/F(\theta^2))\right\}$, $\theta^2 \geq m_B$ and $h(\theta^2, \theta^2) \leq 0$, then there is a unique $\tau^2 \in [m_B, \theta^2]$ such that $h(\tau^2, \theta^2) = 0$, and $p_A^*(\theta) = p_A(\theta; \tau^2, \theta^2)$;
3. if $\text{Gr}(\phi) \cap \left(\text{Gr}(F \cup 1/F)\right) = \left\{(\phi(\theta^2), 1/F(\theta^2))\right\}$, $\theta^2 < m_B$ or $h(\theta^2, \theta^2) > 0$, then there is a unique $\tau^3 \geq \max\left\{m_B, \theta^2\right\}$ such that $h(\tau^3, \tau^3) = 0$, and $p_A^*(\theta) = p_A(\theta; \tau^3, \tau^3)$.

Proof The main idea of the proof is to show that fix a reasonable θ^*, $h(t, \theta^*)$ has a unique zero, which is also the appropriate choice of the lower end of the indifference interval. Note that $h(m_B, \theta^1) > 0$ and $h(\theta^1, \theta^1) < 0$. Subtracting (t, θ^1) from notations, we also have

$$\frac{\partial h}{\partial t} = \left(1 - \int_t^1 p_A(x)dF(z)\right)^{N-1} v_A'(t)$$
$$- \left(F(t) + \int_t^1 p_A(x)dF(x)\right)^{N-1} v_B'(t)$$
$$- \int_{m_A}^t (N-1)\left(1 - \int_x^1 p_A(y)dF(y)\right)^{N-2} p_A(t)f(t)\, dv_A(x)$$
$$- \int_{m_B}^t (N-1)\left(F(x) + \int_x^1 p_A(y)dF(y)\right)^{N-2} (1-p_A(t))f(t)\, dv_B(x)$$

By definition of $p_A(\theta; t, \theta^1)$,

$$\left(1 - \int_t^1 p_A(x)dF(z)\right)^{N-1} v_A'(t) = \left(F(t) + \int_t^1 p_A(x)dF(x)\right)^{N-1} v_B'(t),$$

which implies $\frac{\partial h}{\partial t} < 0$. Therefore, $h(t, \theta^1)$ has a unique zero $\tau^1 \in [m_B, \theta^1]$.

For (2) and (3), there is a minor complication due to the exclusion effect of m_B. We claim that $h(\theta^2, \theta^2) \leq 0$ if and only if $\Delta V(\theta^2) = 0$ (in equilibrium) and clarify when the indifference interval is degenerate. First suppose $\Delta V(\theta^2) = 0$. Let τ^2 be the lower end of the indifference interval. Then

$$V_A(\theta^2) = \int_{m_A}^{\theta^2} \left(1 - \int_x^1 p_A(y; \tau^2, \theta^2)dF(y)\right)^{N-1} dv_A(x)$$
$$\geq \int_{m_A}^{\theta^2} \left(1 - \int_x^1 p_A(y; \theta^2, \theta^2)dF(y)\right)^{N-1} dv_A(x)$$
$$V_B(\theta^2) = \int_{m_B}^{\theta^2} \left(F(x) + \int_x^1 p_A(y; \tau^2, \theta^2)dF(y)\right)^{N-1} dv_B(x)$$
$$\leq \int_{m_B}^{\theta^2} \left(F(x) + \int_x^1 p_A(y; \theta^2, \theta^2)dF(y)\right)^{N-1} dv_B(x)$$

and thus

$h(\theta^2, \theta^2) \leq \Delta V(\theta^2) = 0$. Second suppose $\Delta V(\theta^2) > 0$. Note that we assume that $p_A \equiv 1$ on $[m_A, 1]$ is not an equilibrium. Then by Lemma D.1, there is $\tau^3 \in (\theta^2, 1)$

such that $h(\tau^3, \tau^3) = \Delta V(\tau^3) = 0$. Since $\Delta V' < 0$ on $(\theta^2, 1)$, the indifference interval is degenerate at $\{\theta'\}$. It can be shown by direct calulation that $h(\theta, \theta)$ is decreasing on $(\theta^2, 1)$, and thus $h(\theta^2, \theta^2) > 0$. The argument for existence and uniqueness of τ^2 or τ^3 resembles that for (1), and is omitted here.

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